A COMBINED FEM/BEM DISCRETE NUMERICAL METHOD FOR SOLVING EXTERNAL SCATTERING PROBLEMS IN ACOUSTICS

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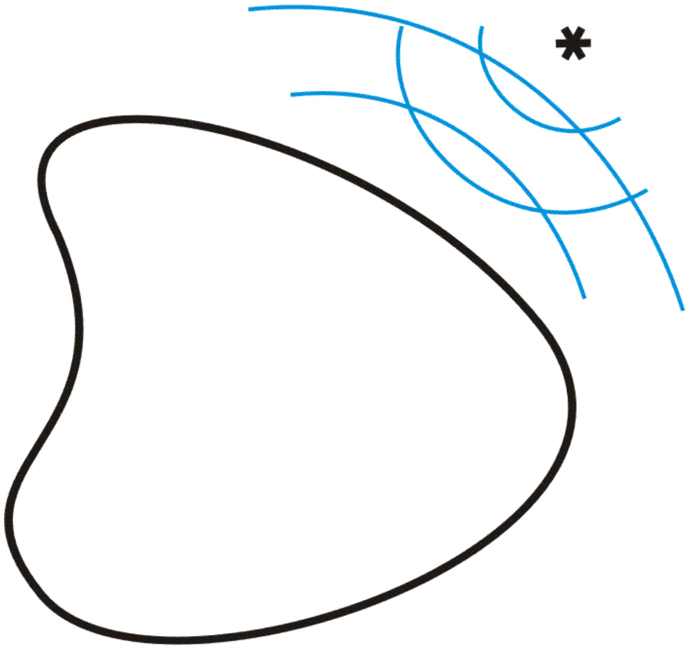
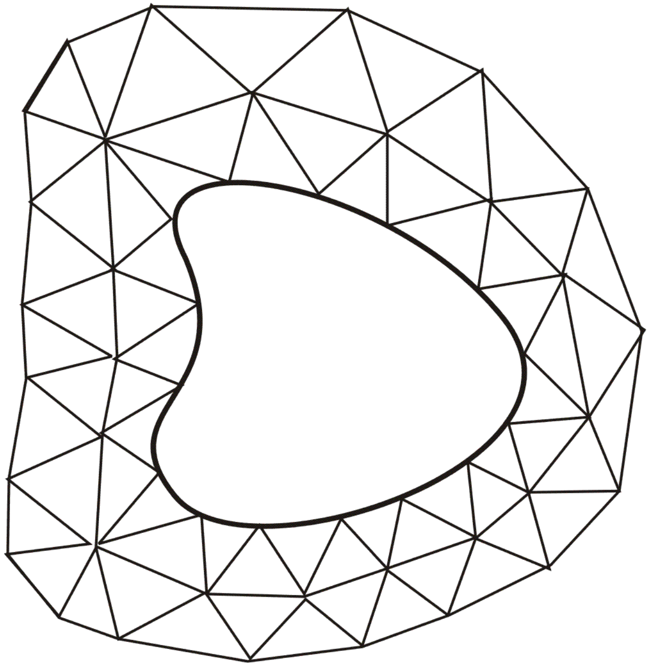
What we would like to solve?

An external diffraction problem (stationary or non-stationary)
with Neumann boundary conditions and some sources



The main dilemma: FEM or BEM?

FEM: Finite Elements Method

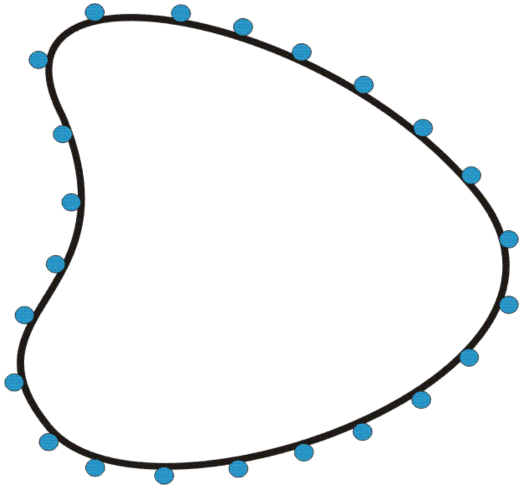


Equation solved: $$\Delta p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = f(r,t)$$

Difficulties:

1. Too many nodes
2. Something should be done at infinity (evacuation of waves), say PML

BEM: Boundary Elements Method



Equation solved:

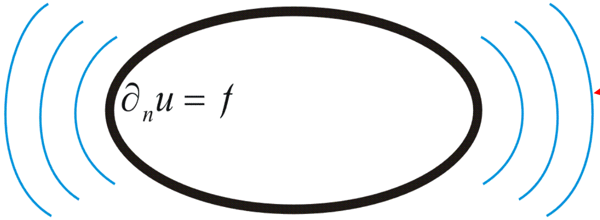
$$\frac{1}{2}u(r) = \int_S \left[G(r,r') f(r') - u(r') \partial_{n'} G(r,r') \right] ds + F(r)$$

(Direct Kirchhoff formulation,
after application Green's theorem to
Helmholtz equation)

Benefits:
1. No problems at infinity
2. Relatively low amount of nodes
3. No dispersion

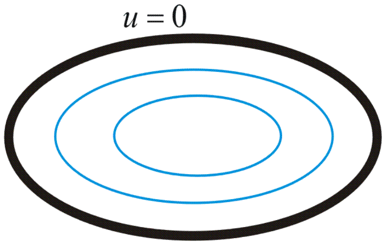
Problems:
1. Singular and oversingular integrals, problems with accuracy
2. Spurious resonances (painful artifacts, see below)
3. Dense matrices

Spurious frequencies (in BEM) and how to kill them



The problem we solve

on spurious frequencies the matrices become ill-conditioned



Spurious frequencies are the eigenvalues of the dual problem

The remedy: CFIE (Burton-Miller) approach

$$\frac{1}{2}u(r)=\int_S\left[G(r,r')f(r')-u(r')\partial_{n'}G(r,r')\right]ds$$

$$+$$

$$\frac{1}{2}f(r)=\int_S\left[\partial_n G(r,r')f(r')-u(r')\partial^2_{n'n}G(r,r')\right]ds$$

$\nu\,\partial_n$

any complex

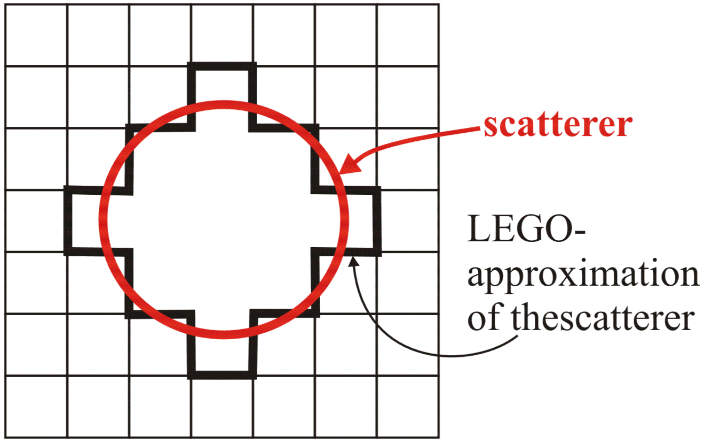
$$\frac{1}{2}u(r)+\int_S\left[u(r')\partial_{n'}G(r,r')-\nu u(r')\partial^2_{n'n}G(r,r')\right]ds=\frac{\nu}{2}f(r)+\int_S\left[G(r,r')f(r')-\nu\,\partial_n G(r,r')f(r')\right]ds$$

painful singularity

New method. Step 1: LEGO-BEM (BAE – Boundary Algebraic Equations)

The main problem of BEM comes from its continuous nature.

If we consider discrete formulation from the very beginning, the Green’s function is not singular (and there should be no integration)



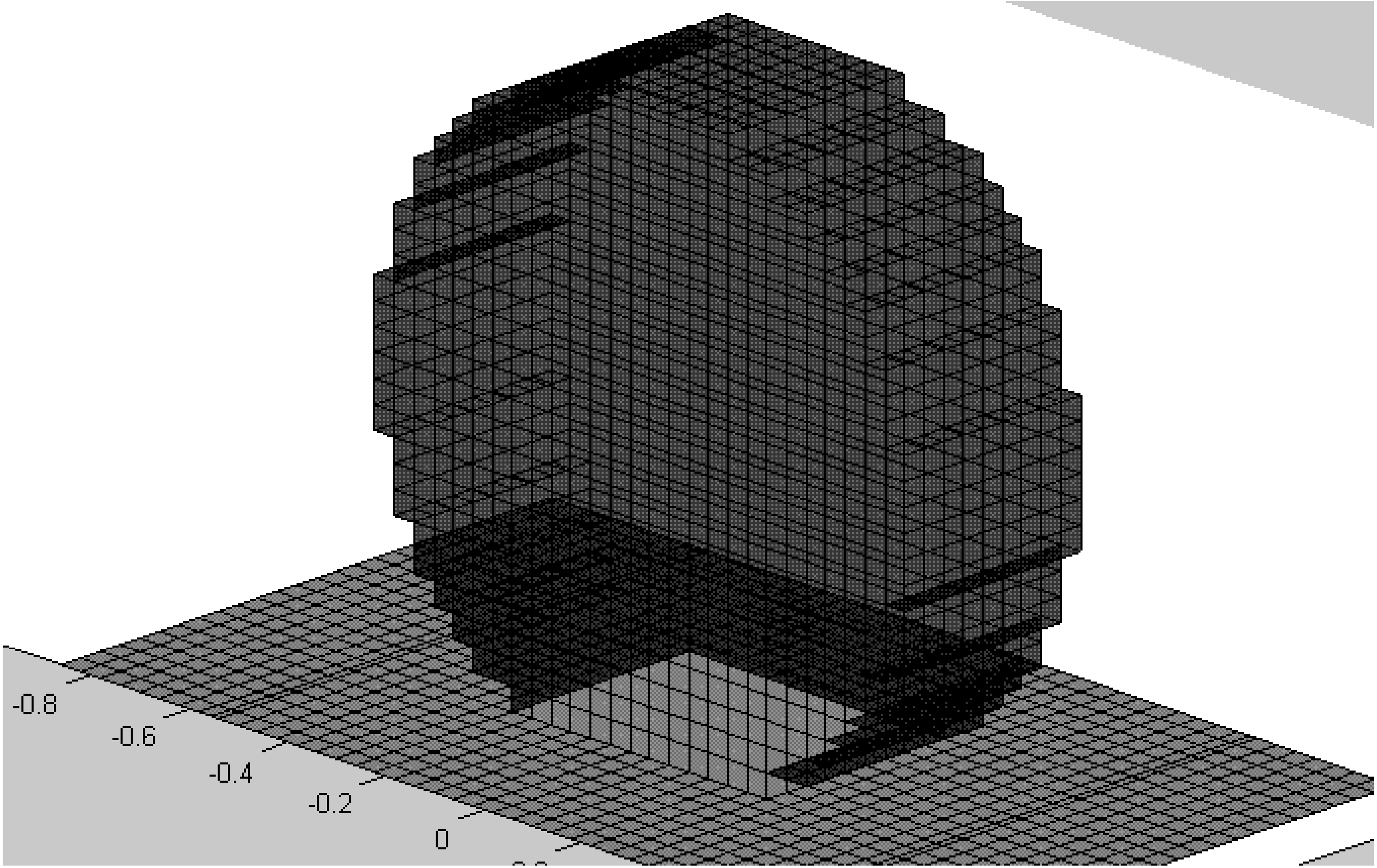
Our aim is to convert Burton-Miller (CFIE) technique to the discrete form

The idea itself appears to be not new ☹

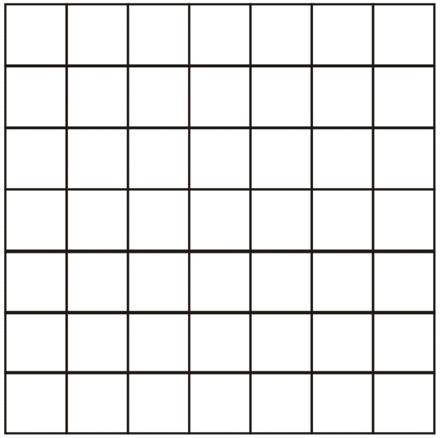
But we were the first to implement an analog of CFIE!

C.Saltzer, 1958;
P-G. Martinsson et al, 2009, 2010
I.Tsukerman, 2005, 2006, 2008, 2011

An example of LEGO mesh



Formalism of the method

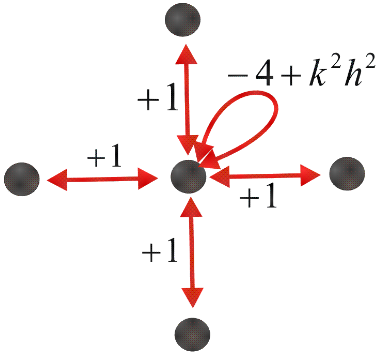

uniform mesh

Equation to solve $\Delta u + k^2 u = f$

Approximate form $\beta_{i,j} u_j = f_i$

Green's function $\beta_{i,j} G_{j,m} = \delta_{m,i}$

example:
5 point scheme
non-zero values
of $\beta_{i,j}$



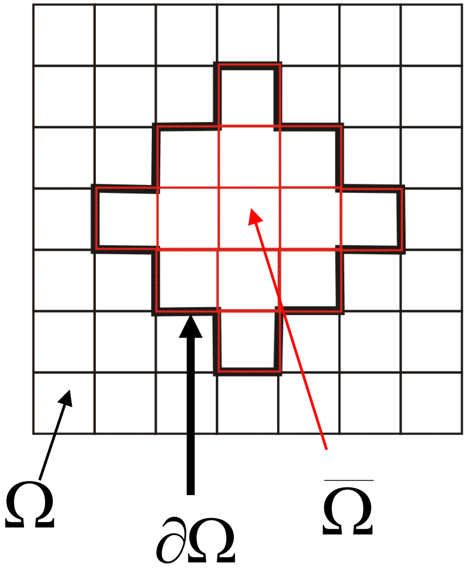
Split $\beta_{i,j}$ between inner and outer mesh:

$$\beta_{i,j} = \beta^I_{i,j} + \beta^O_{i,j}$$

inner

outer

black mesh - outer problem
red mesh - inner problem



Both should be symmetrical: $\beta^I_{i,j} = \beta^I_{j,i}$ $\beta^O_{i,j} = \beta^O_{j,i}$

FEM equation we solve:

$$\sum_{j\in\Omega} \beta^{O}_{i,j} u_j = f_i, \qquad f_i \neq 0 \quad \text{only for} \quad i \in \partial\Omega$$

outer domain Ω

in operator form: $$\beta^{OO} u^{O} = f^{O}$$

BAE (LEGO-BEM) formulation of the problem

$$\sum_{i\in\Omega} \sum_{j\in\partial\Omega} G_{m,i} \beta^{O}_{i,j} u_j = \sum_{j\in\partial\Omega} G_{m,j} f_j \qquad m \in \partial\Omega$$

boundary nodes $\partial\Omega$

in operator form: $$\boxed{G^{BO} \beta^{OB} u^{B} = G^{BB} f^{B}}$$

It may be not clear from the first glance, but this is a direct analogue of the Kirchhoff BEM formulation!

Interpretation of the boundary equation as the NtD (Neumann to Dirichlet) operator

$$G^{BO}\beta^{OB}u^{B} = G^{BB}f^{B}$$

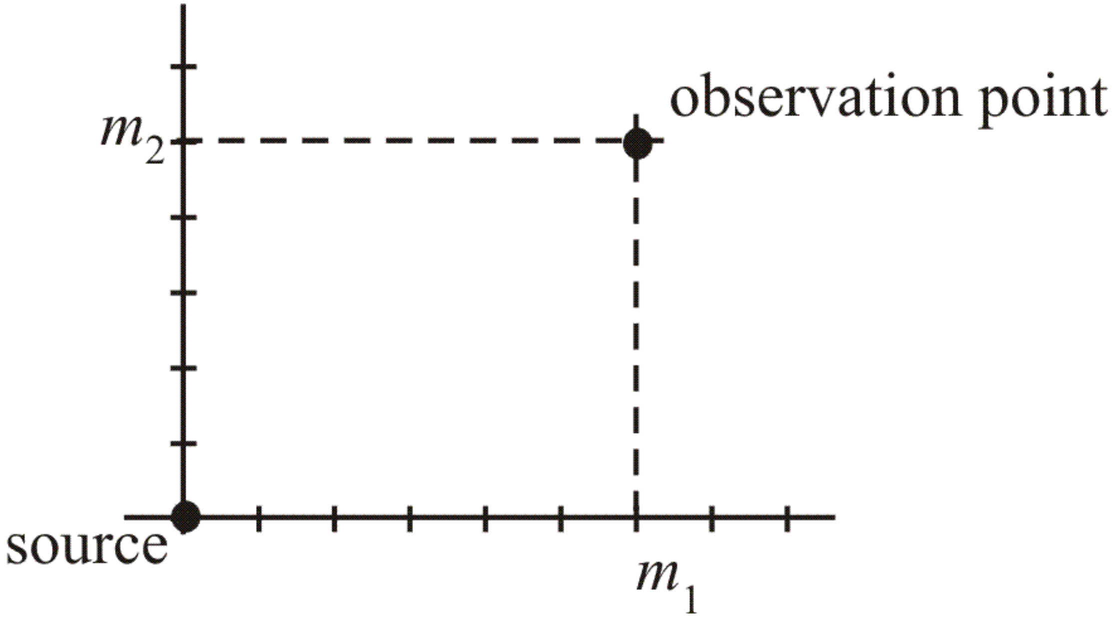
$$f^{B} = \beta^{BO}u^{O}$$

Neumann boundary operator,
discrete analogue of $\partial_n u$

$$\underbrace{u^{B}}_{\text{Dirichlet boundary data}} = \underbrace{(G^{BO}\beta^{OB})^{-1}G^{BB}}_{\text{NtD operator}}\underbrace{(\beta^{BO}u^{O})}_{\text{Neumann boundary data}}$$

<u>This relation guarantees that there are no waves coming from infinity!</u>

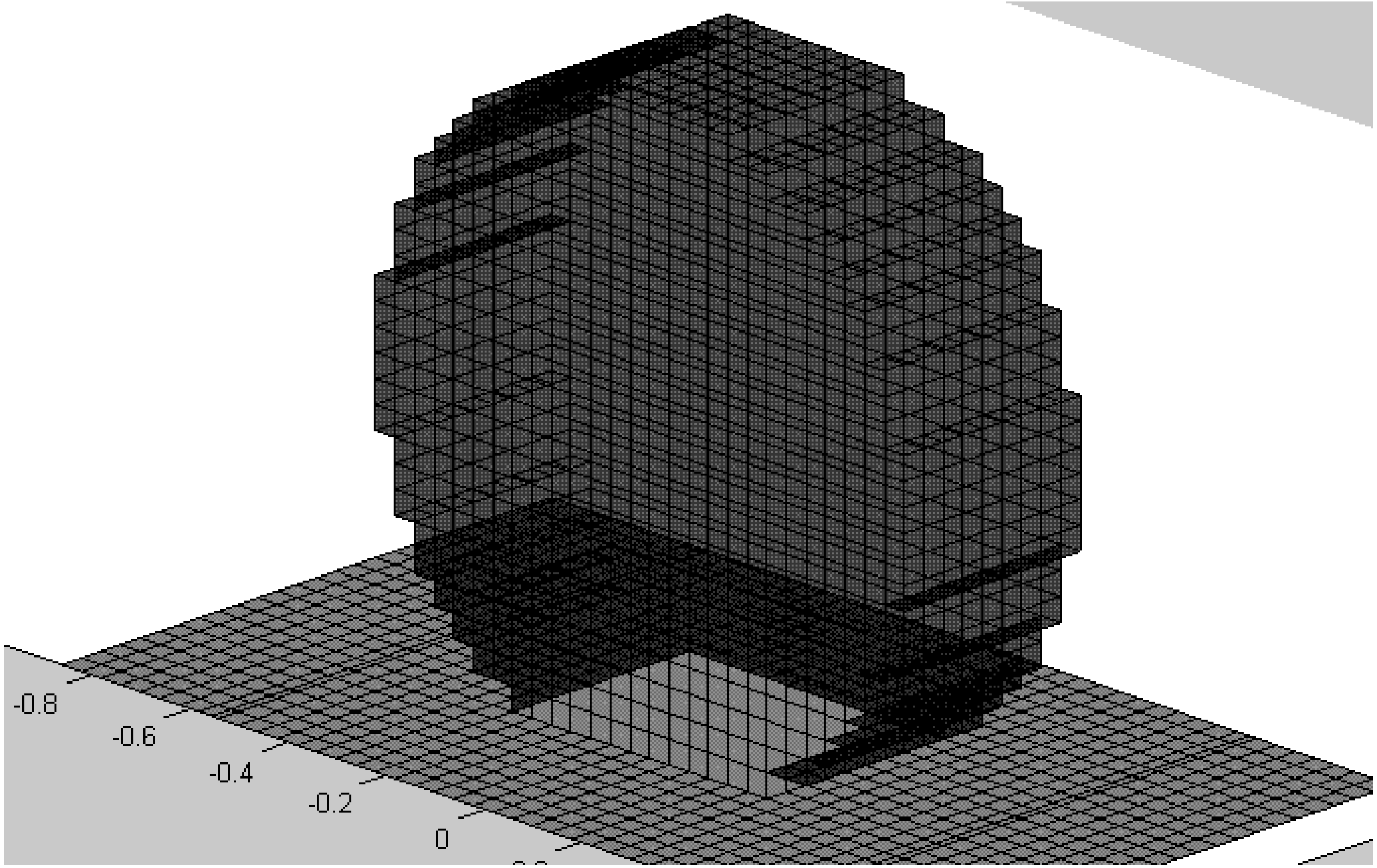
The price we pay: computation (tabulation) of the Green's function



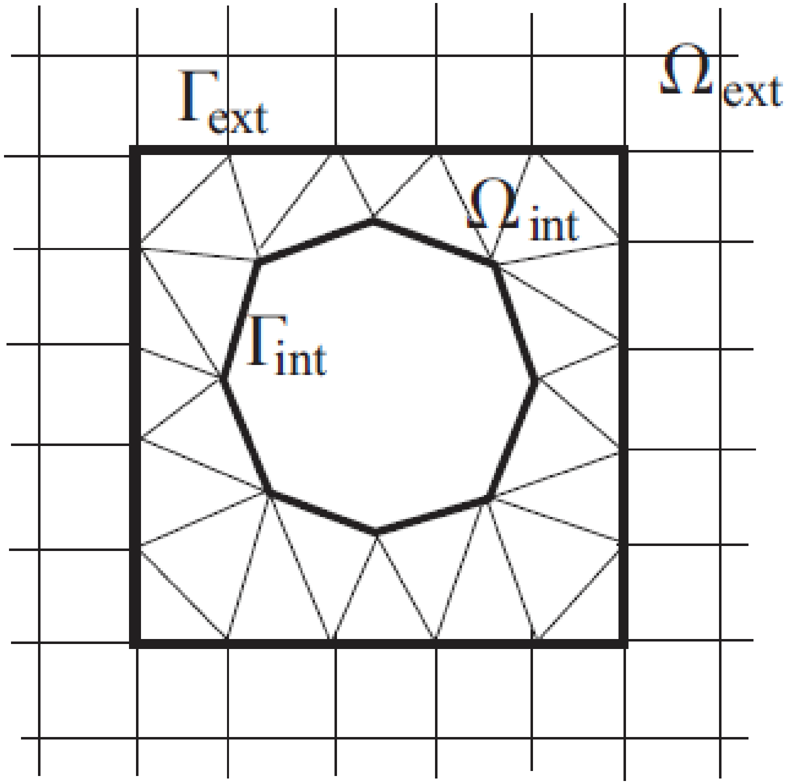
$$G(m_1,m_2)=\frac{h^2}{4\pi i}\int_{-\pi}^{\pi}\frac{\exp\{im_1\xi+i\,|\,m_2\,|\,\Xi(\xi)\}}{\Xi(\xi)}d\xi$$

$$\Xi(\xi)=\arccos(2-\cos\xi-h^2k^2/2)$$

In principle, we can use BAE, but the accuracy is poor due to rough ("LEGO") boundary modeling!



New method, Step 2:
Combining FEM with BAE

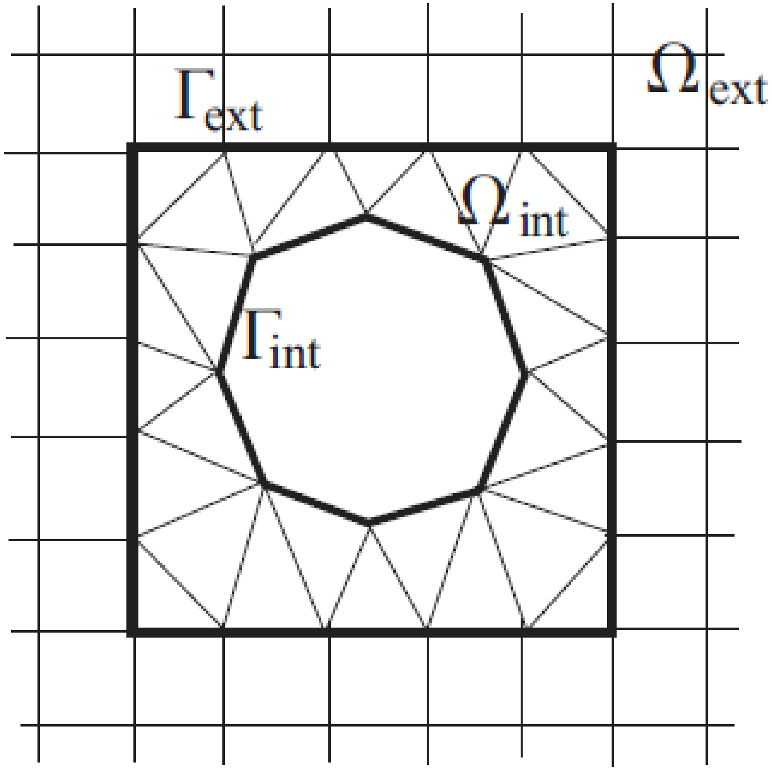


We apply BAE to Ω_{ext}

to find the link between Neumann and Dirichlet data on Γ_{ext}

Then we apply FEM to Ω_{int} using BAE as a boundary condition on Γ_{ext}

Formalism of the method



$I\ :\ \Omega_{\text{int}}$

$O\ :\ \Omega_{\text{ext}}$

$i\ :\ \Omega_{\text{int}} \setminus \Gamma_{\text{ext}}$

$B\ :\ \Gamma_{\text{ext}}$

FEM on Ω_{int} :

$$\beta^{iI} u^{I} = f^{\,i} \qquad \textbf{(*)}$$

Boundary condition on Γ_{ext} :

Note that $(\beta^{BO} + \beta^{BI})u = 0$ on Γ_{ext} (all sources are on Γ_{int})

thus $\beta^{BO} u^{O} = -\beta^{BI} u^{I}$ (Neumann boundary data)

$$u^{B} = -(G^{BO}\beta^{OB})^{-1} G^{BB} (\beta^{BI} u^{I}) \qquad \textbf{(**)}$$

System (*), (**) is what we solve!

Example of mesh Ω_{int}