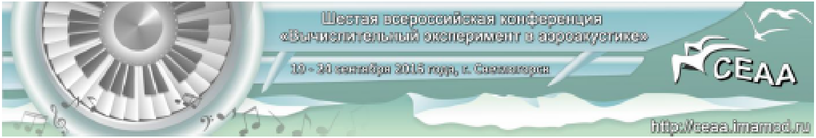


PARALLEL IMPLEMENTATION OF DG METHOD HIGH ORDER OF ACCURACY AND SOLUTION OF CLASSICAL TESTING PROBLEMS

S. Bosnyakov, S. Mikhaylov, V. Podaruev, A. Troshin

The work was done within the framework of the federal target program "Research and development on priority directions of scientific-technological complex of Russia for 2014 - 2020 years." Agreement on the provision of subsidies from the "18" November 2015 № 14.628.21.0005

2016

Introduction

Towards Industrial LES/DNS in Aeronautics -
Paving the Way for Future Accurate CFD

TILDA

Objective: Development and testing of TsAGI code based on the Discontinuous Galerkin Method (DG) of higher-order approximation for the calculation of turbulent flows and distribution of acoustic disturbances

Leader: **A. Wolkov**

In addition, the comparison with the Finite Volume Method (FV) is presented

The work was done within the framework of the federal target program "Research and development on priority directions of scientific-technological complex of Russia for 2014 - 2020 years."Agreement on the provision of subsidies from the "18"November 2015 № 14.628.21.0005

Contents

Contents

DG Method 1/4

The system of equations written in conservative variables:

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = 0.$$

Numerical solution for each of them is a linear combination of basis functions $\varphi_j(\mathbf{x})$:

$$U(\mathbf{x}, t) = \sum_{j=1}^{K_f} u_j(t)\varphi_j(\mathbf{x}),$$

where $u_j(t)$ — expansion coefficients to be determined,

$$\begin{aligned}\{\varphi_j\} = \ & \{1, \\ & \hat{x}, \quad \hat{y}, \quad \hat{z}, \\ & \ldots \\ & \hat{x}^3, \quad \hat{x}^2\hat{y}, \quad \hat{x}^2\hat{z}, \quad \hat{x}\hat{y}^2, \quad \hat{x}\hat{y}\hat{z}, \quad \hat{x}\hat{z}^2, \quad \hat{y}^3, \quad \hat{y}^2\hat{z}, \quad \hat{y}\hat{z}^2, \quad \hat{z}^3\},\end{aligned}$$

where: $\hat{x}, \hat{y}, \hat{z}$ obtained by transition to the coordinate system, the axis which are directed along the ellipsoid of inertia.

After transition these functions are orthonormal using Gram-Schmidt procedure.

DG Method 2/4

The resulting system of equations:

$$\frac{du_i}{dt} + \int\limits_{\Sigma} (\varphi_i \mathbf{F} \cdot \mathbf{n})\, d\Sigma - \int\limits_{\Omega} \nabla(\mathbf{F} \cdot \varphi_i)\, d\Omega = 0, \quad i = \overline{1, K_f}$$

We replace the expression in brackets by means of polynomials $\mathbb{P}(\mathbf{x})$, and use of the Gauss quadrature formula:

$$I_{\hat{\Omega}} = \int\limits_{\hat{\Omega}} \mathbb{P}_n(\xi, \eta, \zeta)\, d\hat{\Omega} = \sum_{i=0}^{N} \omega_i \mathbb{P}_n(\xi_i, \eta_i, \zeta_i),$$

$$I_{\hat{\Sigma}} = \int\limits_{\hat{\Sigma}} \mathbb{P}_m(\xi, \eta)\, d\hat{\Sigma} = \sum_{j=0}^{M} \omega_j \mathbb{P}_m(\xi_j, \eta_j)$$

These formulas are accurate when integrated within the standard elements: cube $\hat{\Omega} = [-1, 1]^3$ and the quadrate of $\hat{\Sigma} = [-1, 1]^2$.

DG Method 3/4

Gauss–Legendre One-dimensional Quadrature

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\sum_{i=1}^{n} w_i f\left(\frac{b-a}{2}\xi_i + \frac{b+a}{2}\right),\; w_i = \frac{2}{(1-\xi_i^2)[\mathcal{L}_n'(\xi_i)]^2},\; n = 1, 2, \dots$$

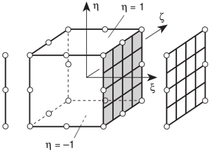
where the numbers ξ_i are n zeros of the Legendre polynomial $\mathcal{L}_n(\xi)$ of order n. These quadrature are accurate for polynomials $\mathbb{P}_{2n-1}(\xi)$ degree at most $2n-1$.
Multidimensional quadrature obtained by Cartesian product of one-dimensional quadrature:

$$\int_{\hat\Sigma} f(\xi,\eta)\,d\hat\Sigma = \sum_{i,j=1}^{n} w_i w_j f(\xi_i,\xi_j), \quad \int_{\hat\Omega} f(\xi,\eta,\zeta)\,d\hat\Omega = \sum_{i,j,k=1}^{n} w_i w_j w_k f(\xi_i,\xi_j,\xi_k)$$

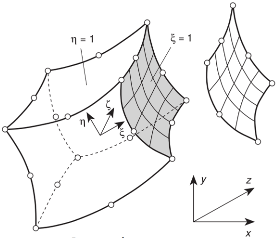
	Optimal number of points			**Maximal number of points**		
N	K=1	K=2	K=3	K=1	K=2	K=3
N_S	4	10	18	25	49	81
N_V	25	38	62	125	343	729
$6N_S + N_V$	49	98	170	275	637	1215
$N/N_{K=1}^{opt}$	1.0	2.0	3.47	5.61	13.0	24.8

DG Method 4/4

To calculate the integrals in the original three-dimensional Cartesian space $\{x, y, z\}$ using Gauss quadrature formulas one should to perform the transition to the local curvilinear coordinate system



Local space



Cartesian space

$$\begin{cases} x \approx \sum\limits_{i=1}^{n} x_i \varphi_i(\xi, \eta, \zeta) \\ y \approx \sum\limits_{i=1}^{n} y_i \varphi_i(\xi, \eta, \zeta) \\ z \approx \sum\limits_{i=1}^{n} z_i \varphi_i(\xi, \eta, \zeta) \end{cases}$$

The desired functions are represented in the form of expansions of basis functions, or functions of the shape, since they determine the form of presentation of the grid elements (linear, quadratic, and so on), where (x_i, y_i, z_i) are coordinate values in the element nodes, $\varphi_i(\xi, \eta, \zeta)$ are the shape functions.

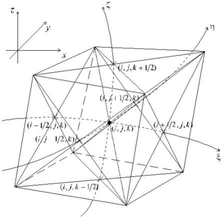
Zienkiewicz O., Taylor R. The Finite Element Method. 1: The Basis. | 5th ed. |

Contents

FV Method 1/2

$$\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}_i}{\partial x_i} = \vec{W}$$

$$\vec{U}^{n+1}_{i,j,k} = \vec{U}^{n}_{i,j,k} - \frac{\tau^n}{V_{i,j,k}} \cdot \left[\left(\vec{F}_{i+1/2} - \vec{F}_{i-1/2} \right) + \left(\vec{F}_{j+1/2} - \vec{F}_{j+1/2} \right) + \left(\vec{F}_{k+1/2} - \vec{F}_{k+1/2} \right) \right] + \tau^n \vec{W}_{i,j,k} .$$

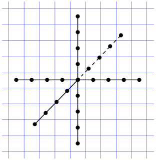


ZEUS EWT TsAGI:

- Multiblock structured grid
- Linear reconstruction + TVD monotonization
- Reconstruction of high order (WENO5, WENO7, WENO9) + MP
- The central difference for diffusion fluxes
- Godunov and Roe schemes for convective fluxes
- Unilateral stencil in the implementation of the boundary conditions

FV Method. More about the WENO. 2/2

1. Implementation of WENO is partial:
   - One-dimensional reconstruction
   - One quadrature point on the side of the cell
   - The central difference for diffusion fluxes



2. In the following tests monotonization is missing (linear weights, no MP)
3. In the following tests Roe fluxes are used:

$$\mathbf{F}_{i+1/2} = \frac{1}{2}[\mathbf{F}(\mathbf{Q}_L) + \mathbf{F}(\mathbf{Q}_R)] - \frac{1}{2}\alpha(A^+ - A^-)(\mathbf{Q}_R - \mathbf{Q}_L);$$

$\alpha = 1 \rightarrow$ upwind scheme,
$\alpha = 0 \rightarrow$ central scheme

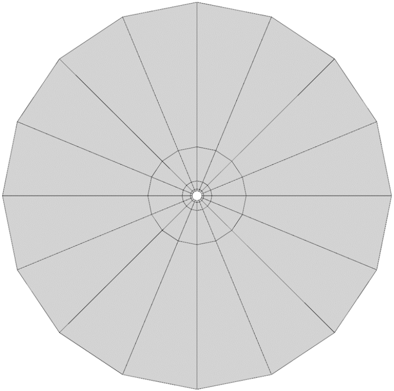
———————

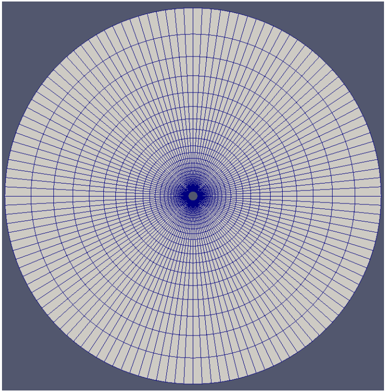
Zhang R., Zhang M., Shu Ch. W. On the order of accuracy and numerical performance of two classes of finite volume WENO schemes. // Communications in Computational Physics, 2011, vol.9, No 3, pp.807-827

Contents

Cylinder. Computational grid and flow parameters

- Series of nested grids with dimension from 16x4x1 to 128x32x1 cells. $R_{cylinder} = 0.5,\ R_{outer} = 20,\ \Delta z = 0.1$
- The cylinder and the plane perpendicular to the cylinder – symmetry
- Outer bound – free-stream
- $p_\infty = 100,000\,Pa,\ T_\infty = 293^\circ K,\ u_\infty = 50\,m/sec\,(M_\infty \approx 0.15)$.

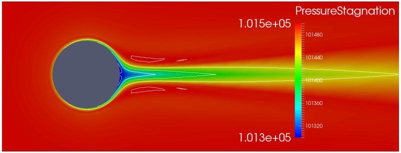




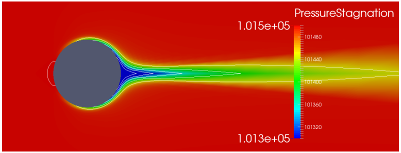
Cylinder. Total pressure flowfield

Grid 128x32x1

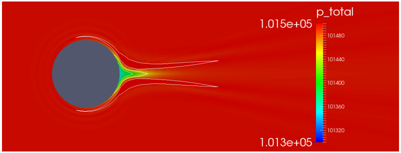
FV, Central



DG, K=1



FV, WENO5



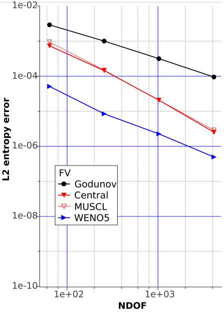
DG, K=3, curvature

Cylinder. Error of entropy in the L2 norm

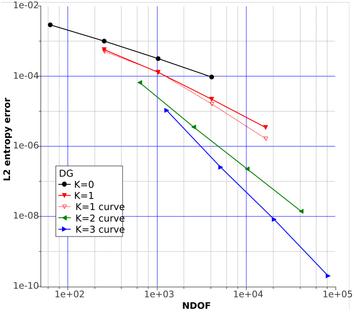
$$e_{entropy} = \left(\frac{p}{p_\infty}\right) \Big/ \left(\frac{\rho}{\rho_\infty}\right)^{\kappa} - 1, \qquad Order = 2\,\frac{\log\left(e_{i-1}\,/\,e_i\right)}{\log\left(NDOF_i\,/\,NDOF_{i-1}\right)}.$$

$NDOF$ = the product of the mesh size and the number of degrees of freedom.

FV



DG



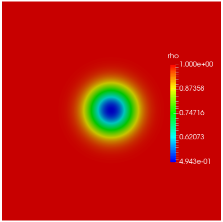
The same result for the drag

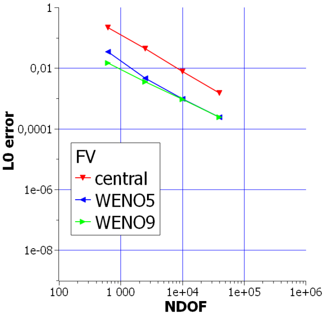
Evolution of 2D vortex. L0 error

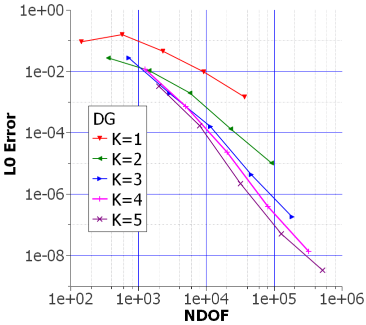
$$u = 1 - \frac{\varepsilon}{2\pi} e^{\frac{1}{2}\left(1-r^2\right)} y,\ v = 1 - \frac{\varepsilon}{2\pi} e^{\frac{1}{2}\left(1-r^2\right)} x,$$

$$T = 1 - \frac{(\gamma-1)\varepsilon^2}{8\gamma\pi^2} e^{\left(1-r^2\right)},\ \frac{p}{\rho^\gamma} = 1,$$

where $r^2 = x^2 + y^2$, и $\varepsilon = 5$





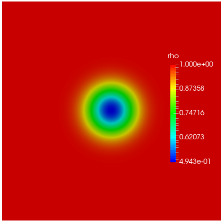


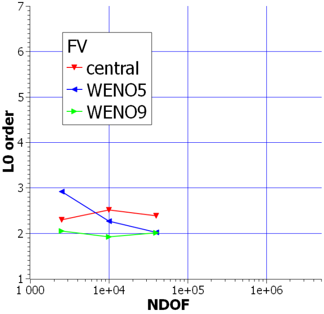
Evolution of 2D vortex. L0 order

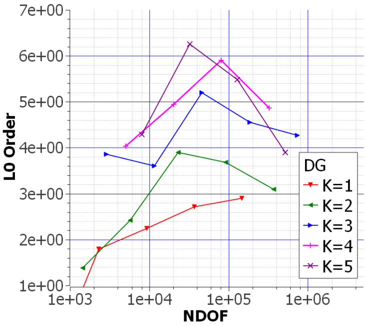
$$u = 1 - \frac{\varepsilon}{2\pi} e^{\frac{1}{2}\left(1-r^2\right)} y, \quad v = 1 - \frac{\varepsilon}{2\pi} e^{\frac{1}{2}\left(1-r^2\right)} x,$$

$$T = 1 - \frac{(\gamma - 1)\varepsilon^2}{8\gamma\pi^2} e^{\left(1-r^2\right)}, \quad \frac{p}{\rho^\gamma} = 1,$$

where $r^2 = x^2 + y^2$, и $\varepsilon = 5$





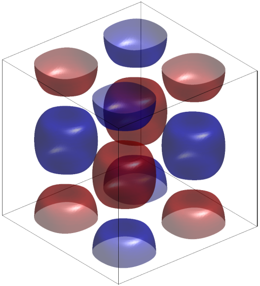


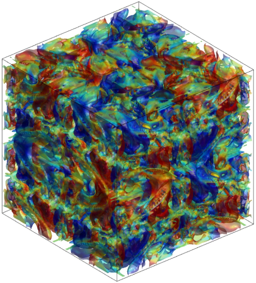
Contents

Taylor-Green Vortex. Base TILDA testcase

$$
\begin{aligned}
u &= V_0 \sin\left(\frac{x}{L}\right)\cos\left(\frac{y}{L}\right)\cos\left(\frac{z}{L}\right) , \\
v &= -V_0 \cos\left(\frac{x}{L}\right)\sin\left(\frac{y}{L}\right)\cos\left(\frac{z}{L}\right) , \\
w &= 0 , \\
p &= p_0 + \frac{\rho_0 V_0^2}{16}\left(\cos\left(\frac{2x}{L}\right) + \cos\left(\frac{2y}{L}\right)\right)\left(\cos\left(\frac{2z}{L}\right) + 2\right)
\end{aligned}
$$

Isosurfaces of pressure (Re = 1600)

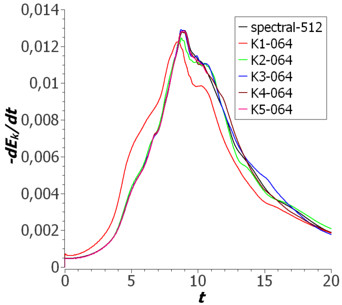


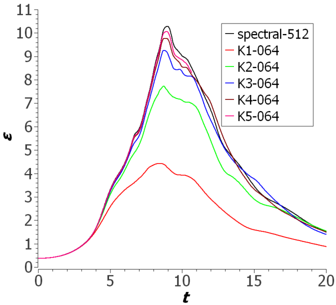


Taylor-Green Vortex. Base TILDA testcase

Mandatory results:

$$E_k = \frac{1}{\rho_0 \Omega} \int\limits_{\Omega} \rho \frac{v \cdot v}{2} d\Omega \qquad \epsilon = \frac{1}{\rho_0 \Omega} \int\limits_{\Omega} \rho \frac{\omega \cdot \omega}{2} d\Omega$$





Comparison with spectral method at fine mesh

————————-

W.M. van Rees, A. Leonard, D.I.Pullin and P. Koumoutsakos, A comparison of vortex and pseudo-spectral methods for the simulation of periodic vortical flows at high Reynolds number, J.Comput.Phys., 230(2011), 2794-2805

Taylor-Green Vortex. Table of calculations. FV

cost – physical time of calculation (16 nodes with 2 x 8 hyperthread cores – totally 512)

error – enstrophy maximum difference obtained in the calculation and in the reference solution

$NDOF = N_{cells}$

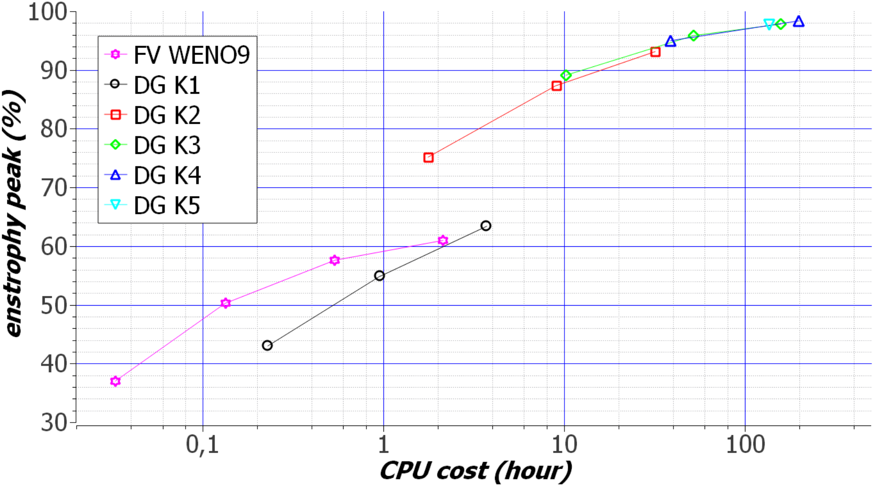
	64^3	96^3	128^3
central			$NDOF = 2.097152$, $cost = 0.311\ h$, $error = 68.31\%$
weno5			$NDOF = 2.097152$, $cost = 0.456\ h$, $error = 45.19\%$
weno9	$NDOF = 262144$, $cost = 0.033\ h$, $error = 63.02\%$	$NDOF = 884736$, $cost = 0.134\ h$, $error = 49.68\%$	$NDOF = 2.097152$, $cost = 0.536\ h$, $error = 42.41\%$

Taylor-Green Vortex. Table of calculations. DG

K – order of polynomial for reconstruction,
bf – number of base functions,
$NDOF = N_{cells} \cdot bf$

K	64^3	96^3	128^3
1 (4 bf)	$NDOF = 1.05 \cdot 10^6$, $cost = 0.23\ h$, $error = 56.98\%$	$NDOF = 3.54 \cdot 10^6$, $cost = 0.96\ h$, $error = 45.13\%$	$NDOF = 8.39 \cdot 10^6$, $cost = 3.72\ h$, $error = 36.67\%$
2 (10 bf)	$NDOF = 2.62 \cdot 10^6$, $cost = 1.77\ h$, $error = 24.87\%$	$NDOF = 8.85 \cdot 10^6$, $cost = 9.06\ h$, $error = 12.61\%$	$NDOF = 21.0 \cdot 10^6$, $cost = 31.97\ h$, $error = 6.90\%$
3 (20 bf)	$NDOF = 5.24 \cdot 10^6$, $cost = 10.25\ h$, $error = 10.05\%$	$NDOF = 17.7 \cdot 10^6$, $cost = 52.04\ h$, $error = 4.17\%$	$NDOF = 41.9 \cdot 10^6$, $cost = 158.7\ h$, $error = 2.18\%$
4 (35 bf)	$NDOF = 9.18 \cdot 10^6$, $cost = 38.95\ h$, $error = 4.97\%$	$NDOF = 31.0 \cdot 10^6$, $cost = 198.0\ h$, $error = 1.66\%$	
5 (56 bf)	$NDOF = 14.7 \cdot 10^6$, $cost = 136.0\ h$, $error = 2.24\%$		

Taylor-Green Vortex

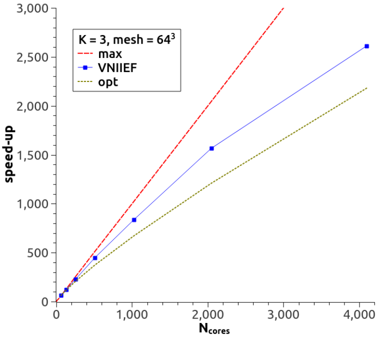


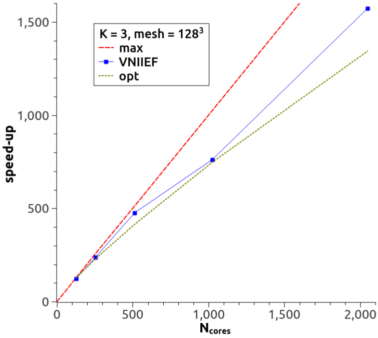
Contents

Taylor-Green Vortex. Scalability

max – maximum possible acceleration $(max = N_{cores})$

opt – acceleration 1.8 times for every doubling of processors





Thank's, IPM, Moscow

Thank's, VNIIEF, Sarov

Questions?