# *A simple robust and accurate a posteriori subcell finite volume limiter for the discontinuous Galerkin method*

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# Objectives

(1) Design a new limiter for the discontinuous Galerkin finite element method that is **simple**, **robust** and **accurate**

(2a) The new limiter **must not destroy** the subcell resolution capability of the DG scheme, neither at discontinuities, nor in smooth regions, where it might have been erroneously activated, or, equivalently

(2b) The limiter must act on a characteristic **length scale** of $h/(N+1)$ and **not** on the length scale $h$ of the main grid, i.e. accuracy improves with $N$ **even at shocks**

(3) The DG limiter should **not** contain **problem-dependent parameters**, like, e.g., the well-known parameter $M$ of the classical TVB limiter of Cockburn and Shu.

(4) The new limiter should work well for **very high** polynomial degrees, say $N$=9.

(5) Ideally, the final DG scheme should become **as robust** as a traditional **second order TVD finite volume scheme**, but **more accurate** on a given computational mesh of characteristic mesh size $h$

# Unlimited Fully Discrete One-Step ADER-DG Scheme

Governing hyperbolic PDE system of the form

$$\frac{\partial \mathbf{Q}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{Q}) = 0 \qquad \text{(PDE)}$$

with the vector of conserved variables **Q** and the nonlinear flux tensor **F**(**Q**). The discrete solution at time $t^n$ is represented by piecewise polynomials of degree $N$ over spatial control volumes $T_i$ as

$$\mathbf{u}_h(\mathbf{x}, t^n) = \sum_l \Phi_l(\mathbf{x}) \hat{\mathbf{u}}_l^n, \quad \mathbf{x} \in T_i \qquad \text{(DG)}$$

Multiplication with a test function $\phi_k$ from the space of piecewise polynomias of degree $N$ and integration over a space-time control volume $T_i$ *x* $[t^n, t^{n+1}]$ yields:

$$\int_{t^n}^{t^{n+1}} \int_{T_i} \Phi_k \frac{\partial \mathbf{Q}}{\partial t} d\mathbf{x}dt + \int_{t^n}^{t^{n+1}} \int_{\partial T_i} \Phi_k \mathbf{F}(\mathbf{Q}) \cdot \mathbf{n}\, dS dt - \int_{t^n}^{t^{n+1}} \int_{T_i} \nabla \Phi_k \cdot \mathbf{F}(\mathbf{Q}) d\mathbf{x}dt = 0$$

# Unlimited Fully Discrete One-Step ADER-DG Scheme

We then introduce the discrete solution (DG) and an **element-local space-time predictor $\mathbf{q}_h(\mathbf{x},t)$,** together with a classical (monotone) numerical flux *G,* as it is used in Godunov-type finite volume schemes.

The fully discrete one-step ADER-DG scheme then simply reads:

$$\left(\int_{T_i} \Phi_k \Phi_l d\mathbf{x}\right)\left(\hat{\mathbf{u}}_l^{n+1} - \hat{\mathbf{u}}_l^n\right) + \int_{t^n}^{t^{n+1}} \int_{\partial T_i} \Phi_k \mathcal{G}\left(\mathbf{q}_h^-, \mathbf{q}_h^+\right) \cdot \mathbf{n}\, dS dt - \int_{t^n}^{t^{n+1}} \int_{T_i} \nabla \Phi_k \cdot \mathbf{F}(\mathbf{q}_h) d\mathbf{x} dt = 0$$

But how to compute the space-time predictor $\mathbf{q}_h(\mathbf{x},t)$, since at the beginning of a time step, only the discrete spatial solution $\mathbf{u}_h(\mathbf{x},t^n)$ at time $t^n$ is known?

Use a weak integral form of the PDE in space-time and solve an element-local Cauchy problem *in the small,* with initial data $\mathbf{u}_h(\mathbf{x},t^n)$, similar to the MUSCL-Hancock scheme or the ENO scheme of Harten et al.

# Element-local Space-time Predictor

Rewrite the governing PDE in a reference coordinate system ξ-τ on a reference element $T_E$:

$$\frac{\partial \mathbf{Q}}{\partial \tau} + \nabla_\xi \cdot \mathbf{F}^*(\mathbf{Q}) = 0, \qquad \mathbf{F}^* := \Delta t \left(\partial \boldsymbol{\xi} / \partial \mathbf{x}\right)^T \cdot \mathbf{F}(\mathbf{Q}).$$

We introduce the two space-time integral operators

$$\langle f, g \rangle = \int_0^1 \int_{T_E} \left( f(\boldsymbol{\xi},\tau) \cdot g(\boldsymbol{\xi},\tau) \right) d\boldsymbol{\xi}\, d\tau, \quad [f, g]^\tau = \int_{T_E} \left( f(\boldsymbol{\xi},\tau) \cdot g(\boldsymbol{\xi},\tau) \right) d\boldsymbol{\xi}$$

The discrete space-time predictor solution and the discrete flux are defined as

$$\mathbf{q}_h = \mathbf{q}_h(\boldsymbol{\xi},\tau) = \sum_l \theta_l(\boldsymbol{\xi},\tau)\hat{\mathbf{q}}_l := \theta_l \hat{\mathbf{q}}_l,$$

**nodal** space-time basis $\theta_l$

$$\mathbf{F}_h^* = \mathbf{F}_h^*(\boldsymbol{\xi},\tau) = \sum_l \theta_l(\boldsymbol{\xi},\tau)\hat{\mathbf{F}}_l^* := \theta_l \hat{\mathbf{F}}_l^*, \qquad \hat{\mathbf{F}}_l^* = \mathbf{F}^*(\hat{\mathbf{q}}_l).$$

# Element-local Space-time Predictor

Multiplication with a **space-time** test function and integration over the space-time reference element $T_E$ x [0,1] yields:

$$\left\langle \theta_k, \frac{\partial \mathbf{q}_h}{\partial \tau} \right\rangle + \left\langle \theta_k, \nabla_\xi \cdot \mathbf{F}_h^*(\mathbf{q}_h) \right\rangle = 0.$$

The initial condition $\mathbf{u}_h(\mathbf{x},t^n)$ is introduced in a **weak sense** after integration by parts **in time** (upwinding in time, causality principle)**:**

$$[\theta_k, \mathbf{q}_h]^1 - [\theta_k, \mathbf{u}_h]^0 - \left\langle \frac{\partial}{\partial \tau}\theta_k, \mathbf{q}_h \right\rangle + \left\langle \theta_k, \nabla_\xi \cdot \mathbf{F}_h^* \right\rangle = 0.$$

The above element-local nonlinear system is easily solved via the following fast-converging fixed-point iteration (discrete Picard iteration):

$$\left( [\theta_k, \theta_l]^1 - \left\langle \frac{\partial}{\partial \tau}\theta_k, \theta_l \right\rangle \right) \hat{\mathbf{q}}_l^{r+1} = [\theta_k, \Phi_l]^0 \hat{\mathbf{u}}_l^n - \langle \theta_k, \nabla_\xi \theta_l \rangle \cdot \mathbf{F}^*(\hat{\mathbf{q}}_l^r),$$

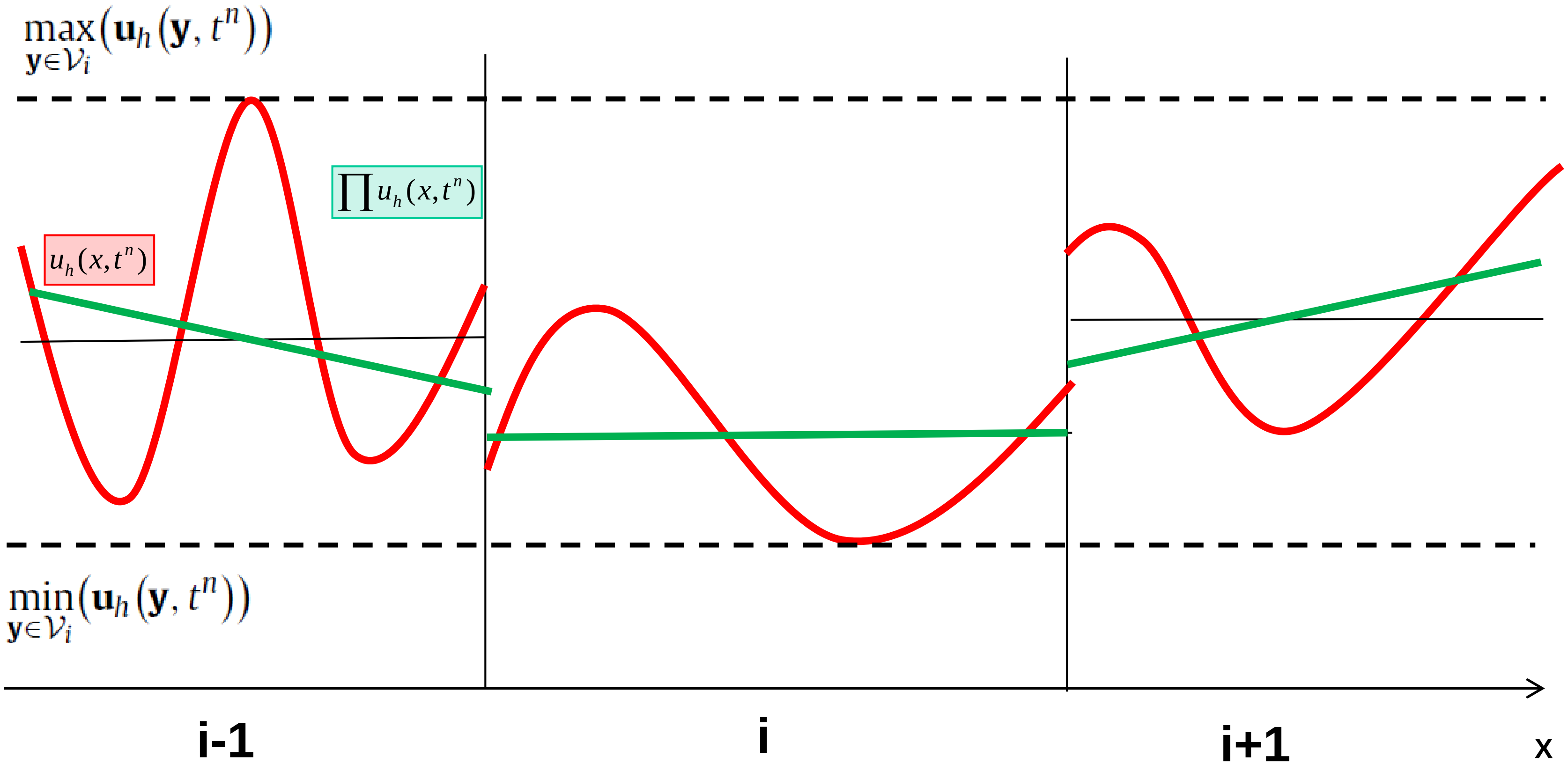
# A new *a posteriori* limiter of DG-FEM methods

- Motivation: develop a **simple, robust** and **parameter-free** limiter for DG that **always works** and which does **not destroy** the **subcell resolution** of DG
- Conventional DG limiters use either artificial viscosity, which needs parameters to be tuned, or nonlinear FV-type reconstruction/limiters (TVB, WENO, HWENO), which **usually destroy** the subcell resolution properties.
- Our new approach: extend the successful *a posteriori* **MOOD** method of Loubère et al., developed in the FV context, also to the DG-FEM framework.
- As very simple *a posteriori* detection criteria, we only use
  - A relaxed discrete maximum principle (**DMP**) in the sense of polynomials
  - **Positivity** of the solution and absence of floating point errors (**NaN**)
- If one of these criteria is violated after a time step, the scheme **goes back** to the old time step and **recomputes** the solution in the troubled cells, using a more robust ADER-WENO or TVD FV scheme **on a fine subgrid** composed of ***2N+1*** subcells per space dimension

# A new *a posteriori* limiter of DG-FEM methods

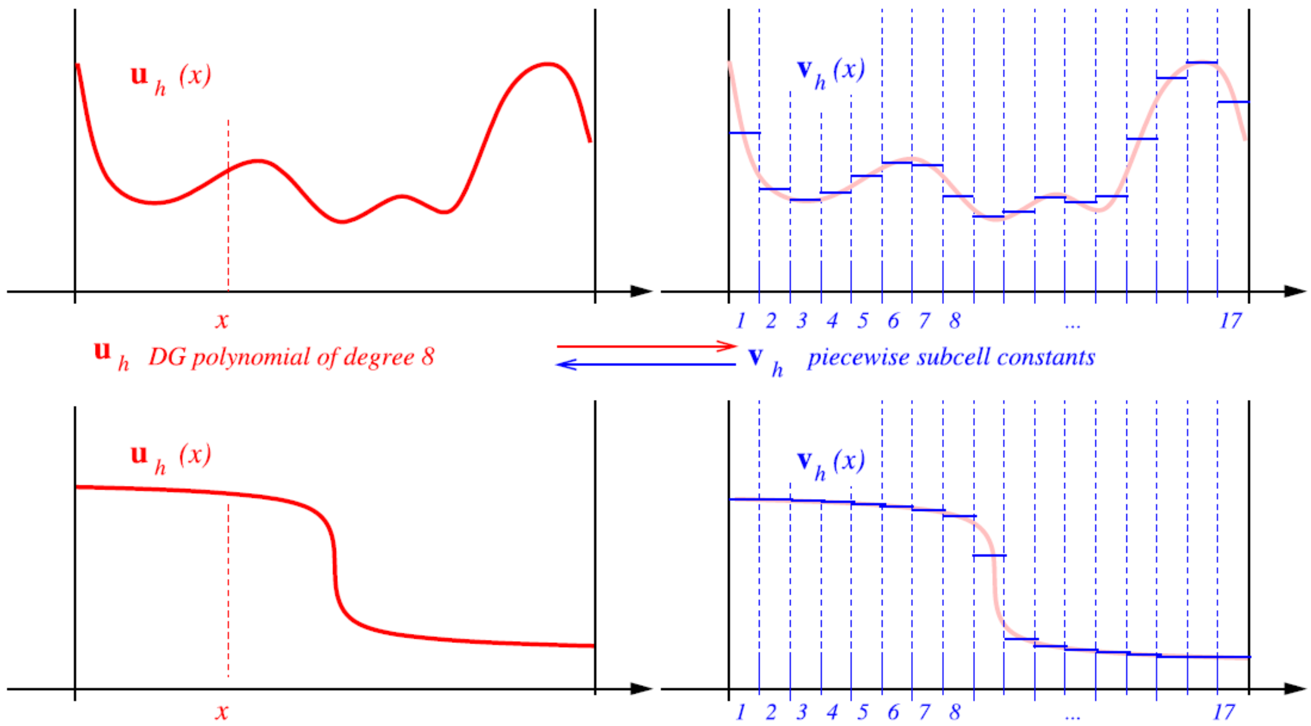
- Classical DG limiters, like WENO/HWENO/slope/moment limiters are based on **nonlinear data post-processing**, while the new DG limiter **recomputes** the discrete solution with a more robust scheme, starting again from a **valid solution** available at the old time level
- Alternative description: dynamic, element-local **checkpointing** and **restarting** of the solver with a more robust scheme on a finer grid
- This enables the limiter even to **cure** floating point errors (**NaN** values appearing after division by zero or after taking roots of negative numbers)
- The new method is by construction **positivity preserving**, **if** the underlying finite volume scheme on the subgrid preserves positivity
- **Local limiter** (in contrast to WENO limiters for DG), since it requires only information from the cell and its direct neighborhood
- As **accurate** as a high order **unlimited DG scheme** in smooth flow regions, but at the same time as **robust** as a **second order TVD scheme** at shocks or other discontinuities, but also at strong rarefactions

# Classical TVB slope/moment limiting of DG




If a classical nonlinear reconstruction-based DG limiter is activated erroneously, there may be important physical information that is lost forever!

# A new *a posteriori* limiter of DG-FEM methods




DG polynomials of degree $N$=8 (left) and **equivalent** data representation on *2N+1*=17 **subcells** (right). Arrows indicate projection (red) and reconstruction (blue)

$$\mathcal{R} \circ \mathcal{P} = \mathcal{I}$$

We use *2N+1* subcells to **match** the DG time step (CFL<*1/(2N+1)*) on the coarse grid with the FV time step (CFL<1) on the fine subgrid.

# A new *a posteriori* limiter of DG-FEM methods

Projection from the DG polynomials to the subcell averages

$$\mathbf{v}_{i,j}^n = \frac{1}{|S_{i,j}|} \int_{S_{i,j}} \mathbf{u}_h(\mathbf{x}, t^n) d\mathbf{x} = \frac{1}{|S_{i,j}|} \int_{S_{i,j}} \phi_l(\mathbf{x}) d\mathbf{x}\, \hat{\mathbf{u}}_l^n, \quad \forall S_{i,j} \in \mathcal{S}_i.$$

Reconstruction of DG polynomials from the subcell averages

$$\int_{S_{i,j}} \mathbf{u}_h(\mathbf{x}, t^n) d\mathbf{x} = \int_{S_{i,j}} \mathbf{v}_h(\mathbf{x}, t^n) d\mathbf{x}, \quad \forall S_{i,j} \in \mathcal{S}_i.$$

$$\int_{T_i} \mathbf{u}_h(\mathbf{x}, t^n) d\mathbf{x} = \int_{T_i} \mathbf{v}_h(\mathbf{x}, t^n) d\mathbf{x}.$$

Linear constraint: conservation

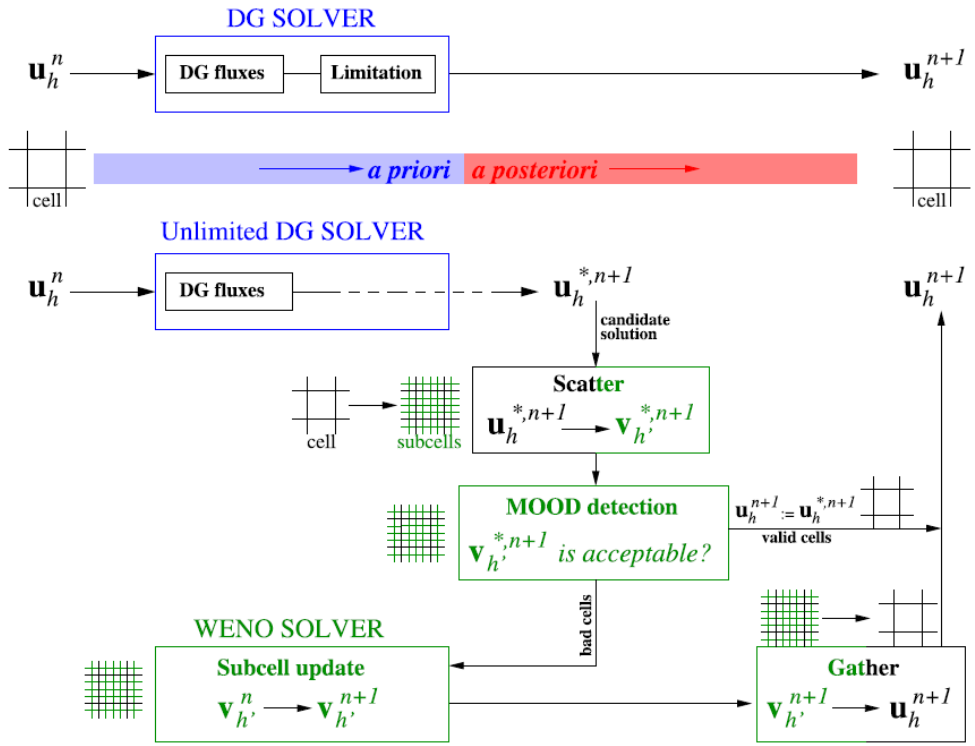
Overdetermined system, solved by a constrained LSQ algorithm.

# *A posteriori* detection criteria and DG-MOOD flowchart

**Positivity:** $\pi_k\left(\mathbf{u}_h^*(\mathbf{x}, t^{n+1})\right) > 0,$

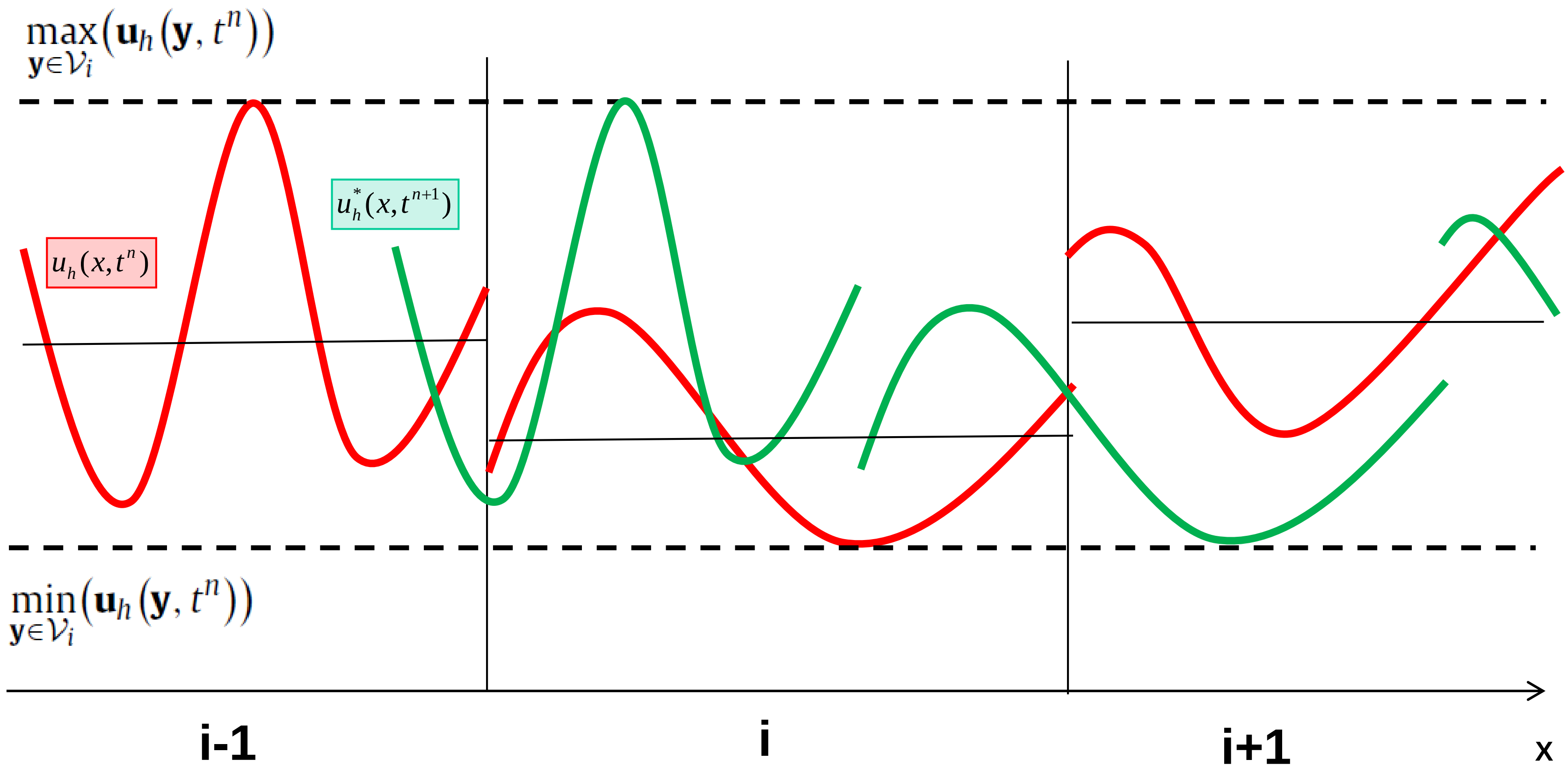
**Relaxed DMP in the sense of polynomials:**

$$\min_{\mathbf{y}\in\mathcal{V}_i}\left(\mathbf{u}_h(\mathbf{y}, t^n)\right) - \delta \le \mathbf{u}_h^*(\mathbf{x}, t^{n+1}) \le \max_{\mathbf{y}\in\mathcal{V}_i}\left(\mathbf{u}_h(\mathbf{y}, t^n)\right) + \delta,$$



# DMP in the sense of polynomials

$$\min_{\mathbf{y}\in\mathcal{V}_i}\left(\mathbf{u}_h\left(\mathbf{y},t^n\right)\right)-\delta\leq\mathbf{u}_h^*\left(\mathbf{x},t^{n+1}\right)\leq\max_{\mathbf{y}\in\mathcal{V}_i}\left(\mathbf{u}_h\left(\mathbf{y},t^n\right)\right)+\delta,$$



# Summary of the ADER-DG-MOOD scheme

Verification of the DMP and the positivity on the **candidate solution $\mathbf{u}_h^*(\mathbf{x},t^{n+1})$:**

$$\min_{\mathbf{y}\in\mathcal{V}_i}\left(\mathbf{v}_h\left(\mathbf{y},t^n\right)\right)-\delta\le\mathbf{v}_h^*\left(\mathbf{x},t^{n+1}\right)\le\max_{\mathbf{y}\in\mathcal{V}_i}\left(\mathbf{v}_h\left(\mathbf{y},t^n\right)\right)+\delta,\quad\forall\mathbf{x}\in T_i,$$

$$\pi_k\left(\mathbf{u}_h^*\left(\mathbf{x},t^{n+1}\right)\right)>0,\quad\forall\mathbf{x}\in T_i,\ \forall k,$$

If a cell does not satisfy both criteria, flag it as troubled cell, $\beta_i^{n+1}=1$, **<u>discard</u>** the DG solution and **<u>recompute</u>** it with a more robust third order **ADER-WENO** or an even more robust **second order TVD finite** volume scheme on the **<u>fine</u> <u>subgrid</u>**:

$$\mathbf{v}_h\left(\mathbf{x},t^{n+1}\right)=\mathcal{A}\left(\mathbf{v}_h\left(\mathbf{x},t^n\right)\right)$$

$$\mathbf{v}_h(\mathbf{x},t^n)=\begin{cases}\mathcal{P}\left(\mathbf{u}_h(\mathbf{x},t^n)\right) & \text{if}\quad\beta_j^n=0,\\ \mathcal{A}(\mathbf{v}_h(\mathbf{x},t^{n-1})) & \text{if}\quad\beta_j^n=1.\end{cases}\qquad\mathbf{x}\in T_j\qquad\forall T_j\in\mathcal{V}_i.$$

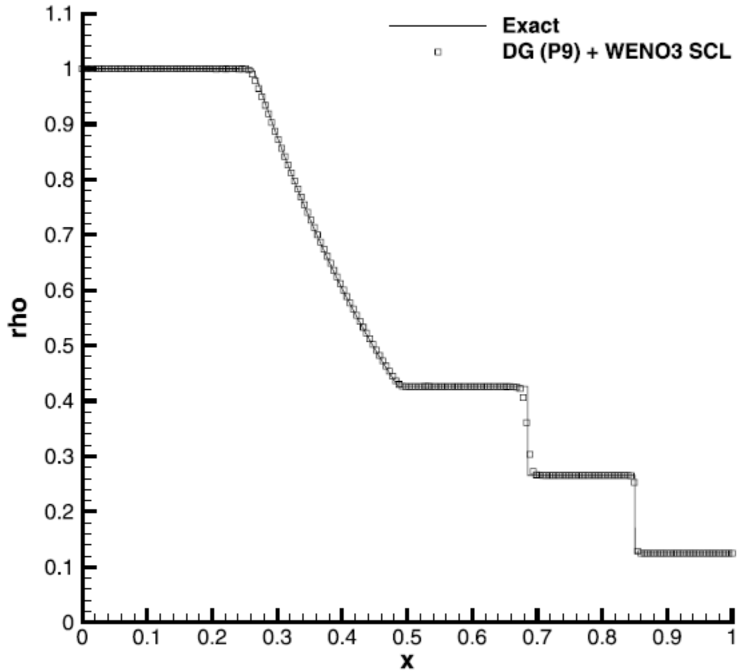
Finally, **reconstruct** the DG polynomial from the subcell averages:

$$\mathbf{u}_h\left(\mathbf{x},t^{n+1}\right)=\mathcal{R}\left(\mathbf{v}_h\left(\mathbf{x},t^{n+1}\right)\right)\qquad\text{or}\qquad\mathbf{u}_h(\mathbf{x},t^{n+1})=\mathcal{R}\left(\mathcal{A}(\mathbf{v}_h(\mathbf{x},t^n))\right)$$

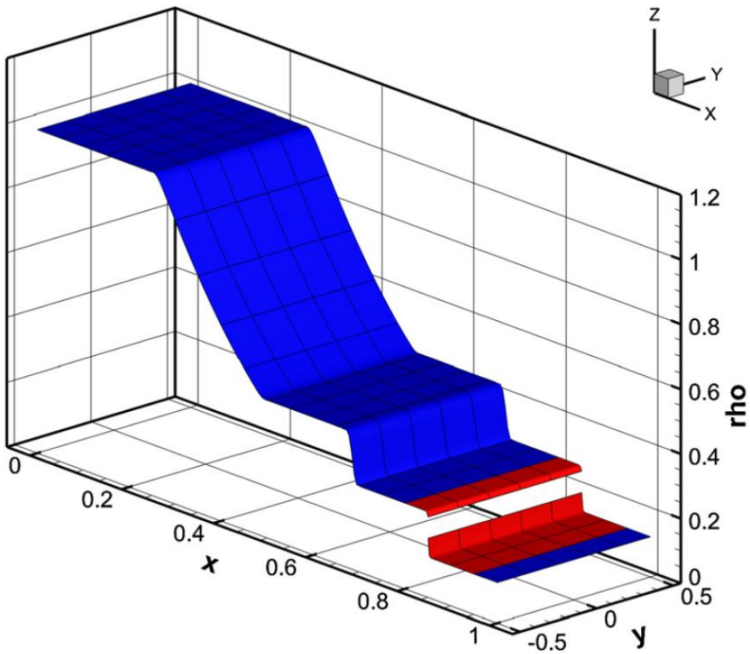
# 2D Numerical Convergence Results P2-P9 (Euler)

| | $N_x$ | $L^1$ error | $L^2$ error | $L^\infty$ error | $L^1$ order | $L^2$ order | $L^\infty$ order | Theor. |
|---|---|---|---|---|---|---|---|---|
| DG-$\mathbb{P}_2$ | 25 | 9.33E−03 | 2.07E−03 | 2.02E−03 | – | – | – | 3 |
| | 50 | 6.70E−04 | 1.58E−04 | 1.66E−04 | 3.80 | 3.71 | 3.60 | |
| | 75 | 1.67E−04 | 4.07E−05 | 4.45E−05 | 3.43 | 3.35 | 3.25 | |
| | 100 | 6.74E−05 | 1.64E−05 | 1.82E−05 | 3.15 | 3.15 | 3.10 | |
| DG-$\mathbb{P}_3$ | 25 | 5.77E−04 | 9.42E−05 | 7.84E−05 | – | – | – | 4 |
| | 50 | 2.75E−05 | 4.52E−06 | 4.09E−06 | 4.39 | 4.38 | 4.26 | |
| | 75 | 4.36E−06 | 7.89E−07 | 7.55E−07 | 4.55 | 4.30 | 4.17 | |
| | 100 | 1.21E−06 | 2.37E−07 | 2.38E−07 | 4.46 | 4.17 | 4.01 | |
| DG-$\mathbb{P}_4$ | 20 | 1.54E−04 | 2.18E−05 | 2.20E−05 | – | – | – | 5 |
| | 30 | 1.79E−05 | 2.46E−06 | 2.13E−06 | 5.32 | 5.37 | 5.75 | |
| | 40 | 3.79E−06 | 5.35E−07 | 5.18E−07 | 5.39 | 5.31 | 4.92 | |
| | 50 | 1.11E−06 | 1.61E−07 | 1.46E−07 | 5.50 | 5.39 | 5.69 | |
| DG-$\mathbb{P}_5$ | 10 | 9.72E−04 | 1.59E−04 | 2.00E−04 | – | – | – | 6 |
| | 20 | 1.56E−05 | 2.13E−06 | 2.14E−06 | 5.96 | 6.22 | 6.55 | |
| | 30 | 1.14E−06 | 1.64E−07 | 1.91E−07 | 6.45 | 6.33 | 5.96 | |
| | 40 | 2.17E−07 | 2.97E−08 | 3.59E−08 | 5.77 | 5.93 | 5.82 | |
| DG-$\mathbb{P}_6$ | 5 | 2.24E−02 | 4.15E−03 | 3.11E−03 | – | – | – | 7 |
| | 10 | 1.76E−04 | 2.75E−05 | 2.86E−05 | 6.99 | 7.24 | 6.76 | |
| | 20 | 1.67E−06 | 2.28E−07 | 2.26E−07 | 6.72 | 6.91 | 6.98 | |
| | 25 | 3.60E−07 | 4.96E−08 | 6.27E−08 | 6.86 | 6.84 | 5.74 | |
| DG-$\mathbb{P}_7$ | 5 | 5.50E−03 | 1.22E−03 | 1.46E−03 | – | – | – | 8 |
| | 10 | 4.63E−05 | 6.26E−06 | 6.95E−06 | 6.89 | 7.61 | 7.71 | |
| | 15 | 1.62E−06 | 2.20E−07 | 2.29E−07 | 8.28 | 8.26 | 8.42 | |
| | 20 | 2.05E−07 | 2.80E−08 | 2.28E−08 | 7.18 | 7.17 | 8.01 | |
| DG-$\mathbb{P}_8$ | 4 | 9.11E−03 | 1.80E−03 | 3.44E−03 | – | – | – | 9 |
| | 8 | 4.97E−05 | 7.51E−06 | 6.93E−06 | 7.52 | 7.90 | 8.96 | |
| | 10 | 7.50E−06 | 1.05E−06 | 1.18E−06 | 8.47 | 8.81 | 7.95 | |
| | 15 | 2.40E−07 | 3.34E−08 | 3.09E−08 | 8.49 | 8.51 | 8.98 | |
| DG-$\mathbb{P}_9$ | 4 | 3.95E−03 | 7.89E−04 | 1.42E−03 | – | – | – | 10 |
| | 8 | 1.01E−05 | 1.44E−06 | 1.52E−06 | 8.61 | 9.09 | 9.87 | |
| | 10 | 1.44E−06 | 2.00E−07 | 2.27E−07 | 8.74 | 8.85 | 8.51 | |
| | 12 | 2.67E−07 | 3.70E−08 | 3.77E−08 | 9.26 | 9.25 | 9.85 | |

# ADER-DG-MOOD Results

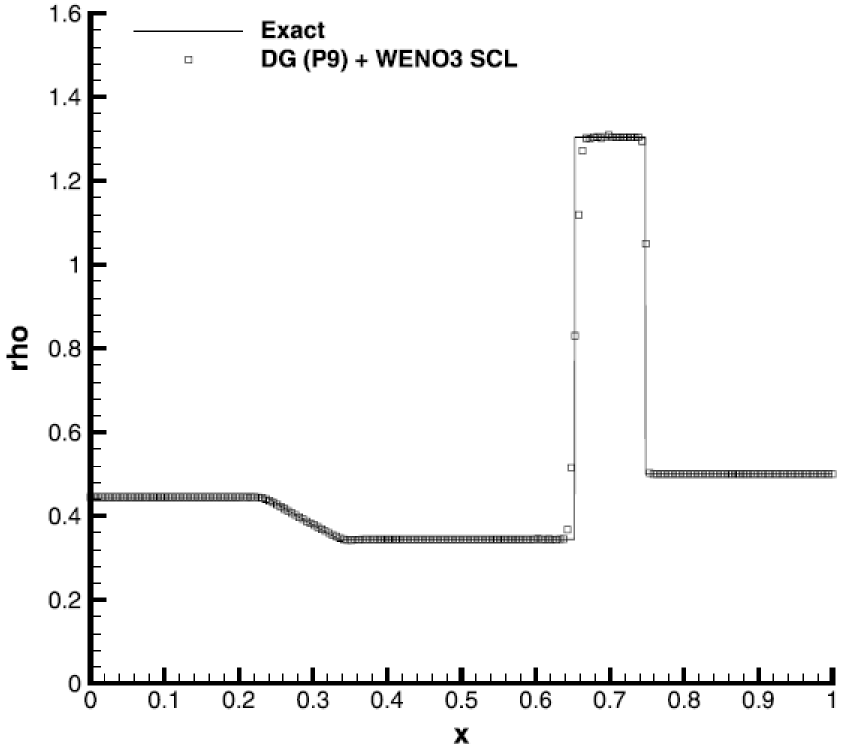



**Sod shock tube,**
**20x5 elements (N=9)**

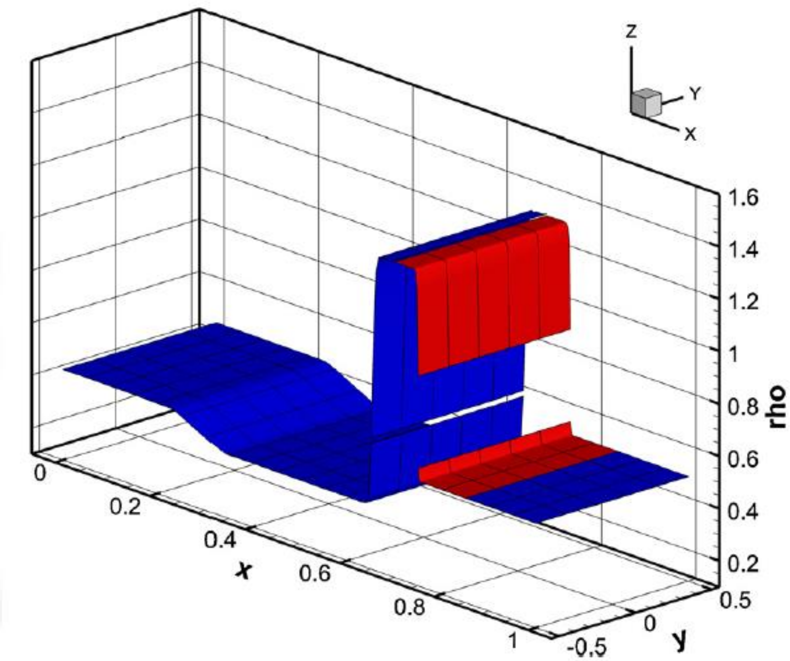



**Limited cells (red),**
**Unlimited cells (blue)**

# ADER-DG-MOOD Results

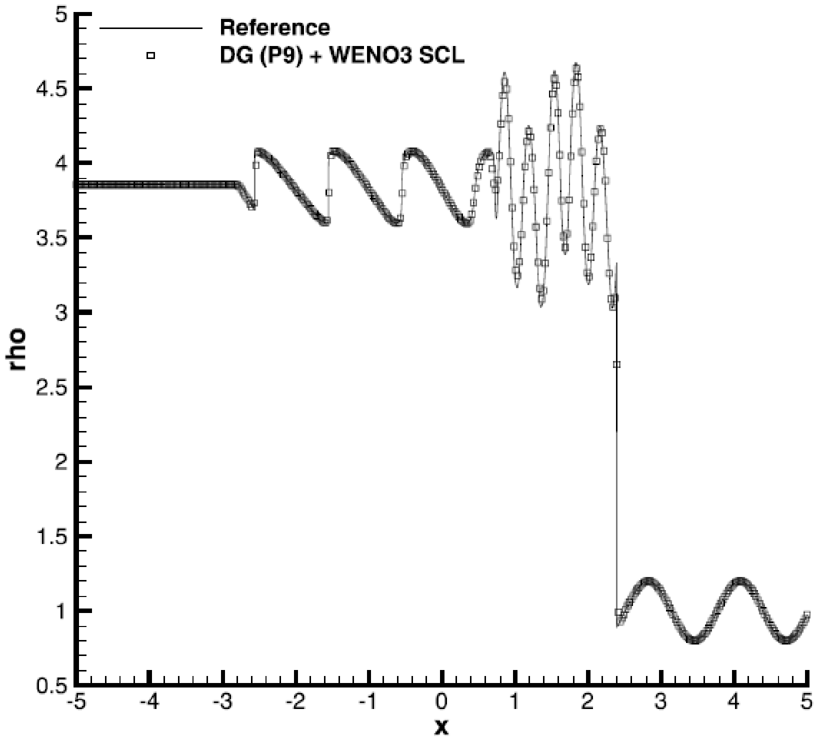


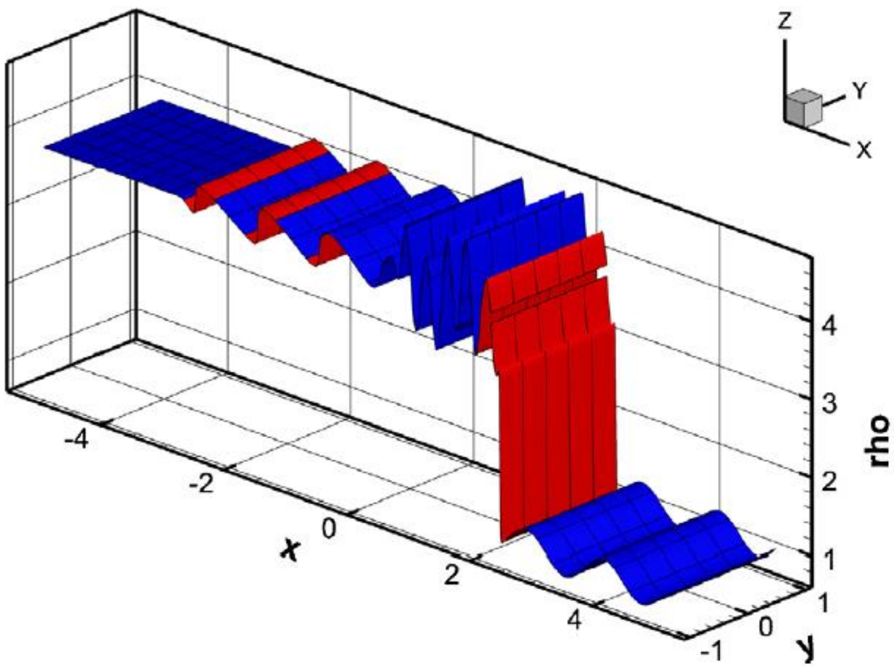

**Lax shock tube,**
**20x5 elements (N=9)**




**Limited cells (red),**
**Unlimited cells (blue)**

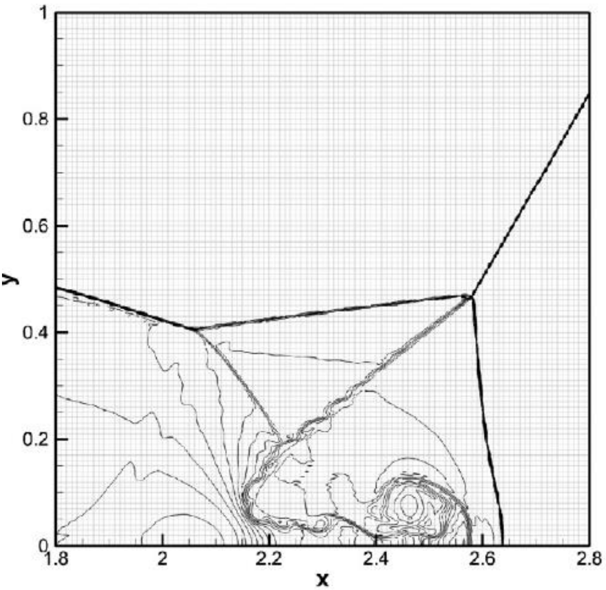
# ADER-DG-MOOD Results

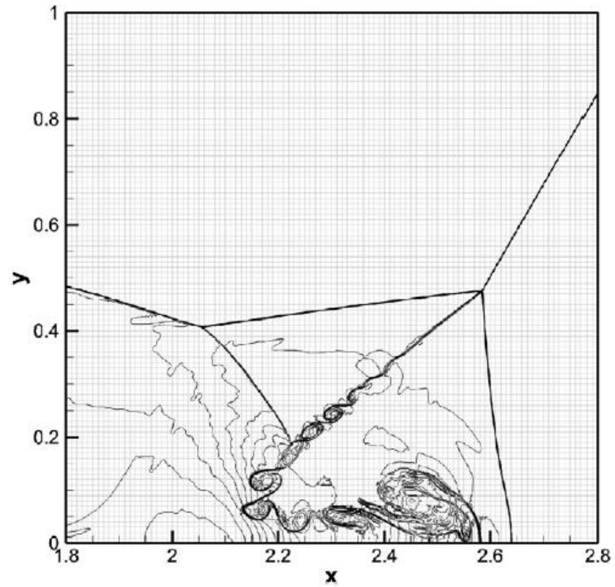


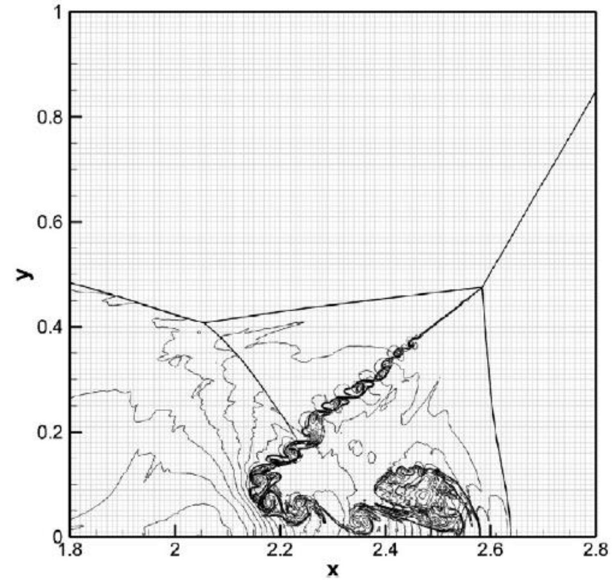



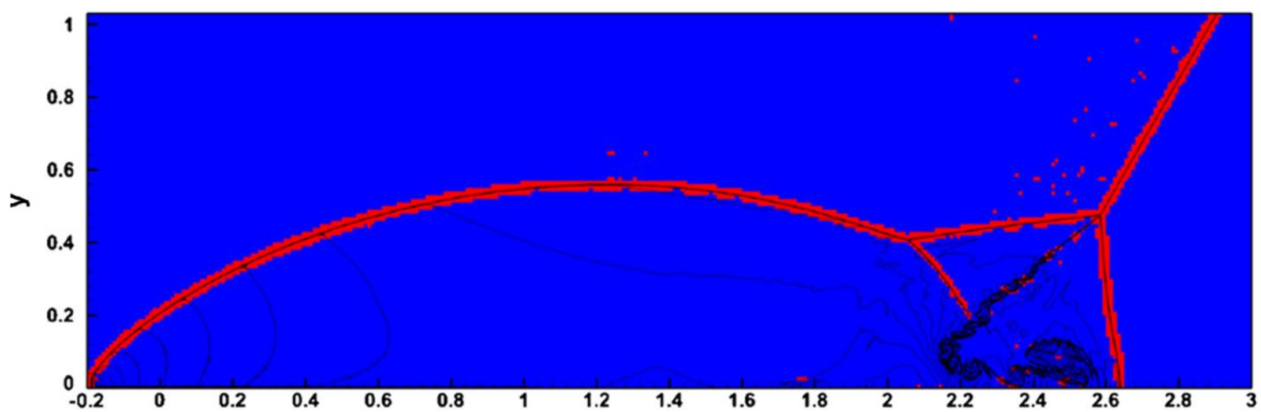

**Shock-density interaction problem of Shu & Osher**
**40x5 cells (N=9). Unlimited cells (blue) and limited cells (red)**

# Double Mach Reflection Problem



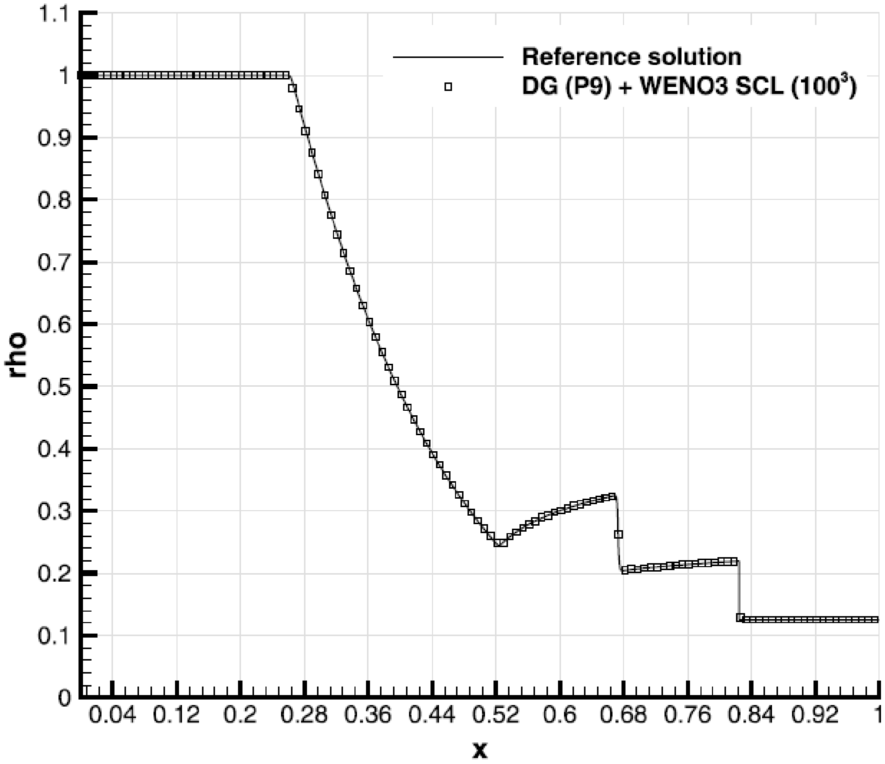



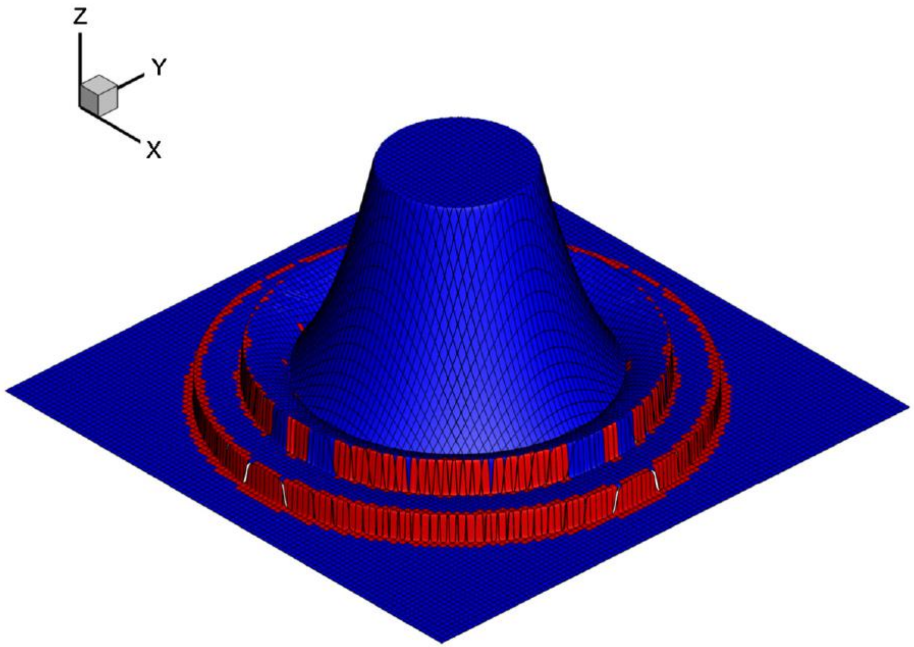




**300x100 cells (N=2, 5, 9). Unlimited cells (blue) and limited cells (red)**

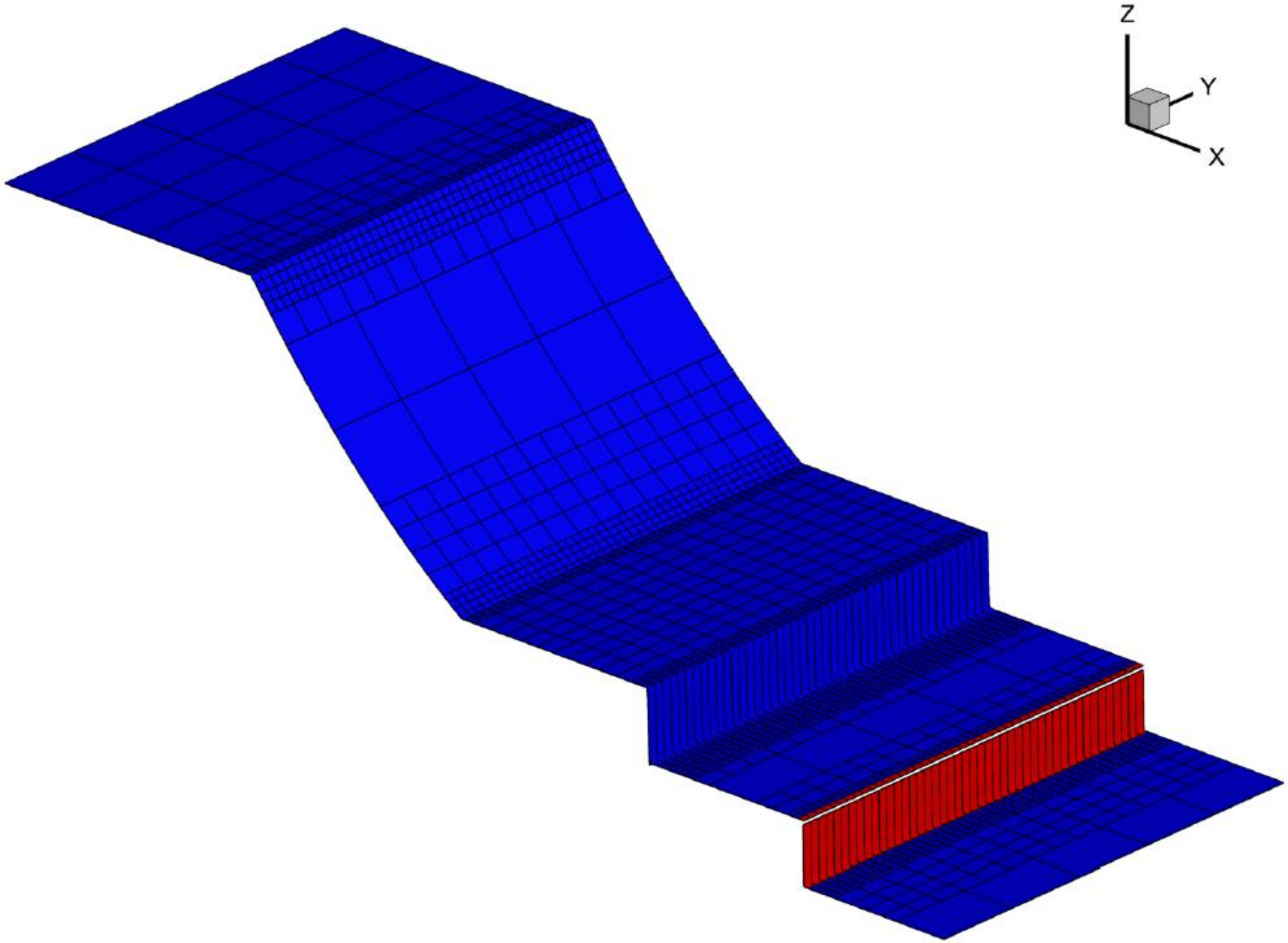
# 3D Spherical Explosion Problem



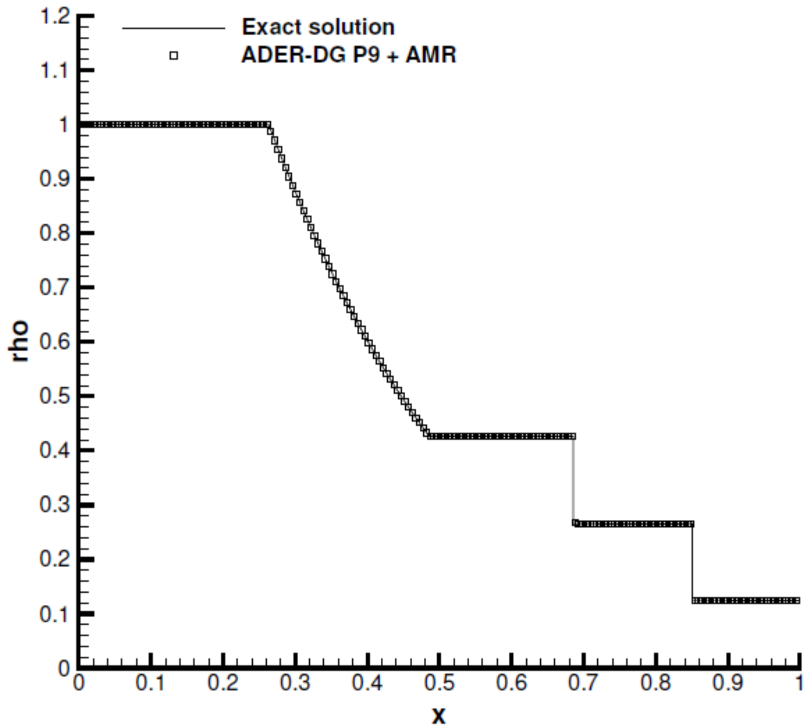

**$100^3$ cells (N=9), corresponding to 10 billion space-time degrees of freedom per time step. Unlimited cells (blue) and limited cells (red)**

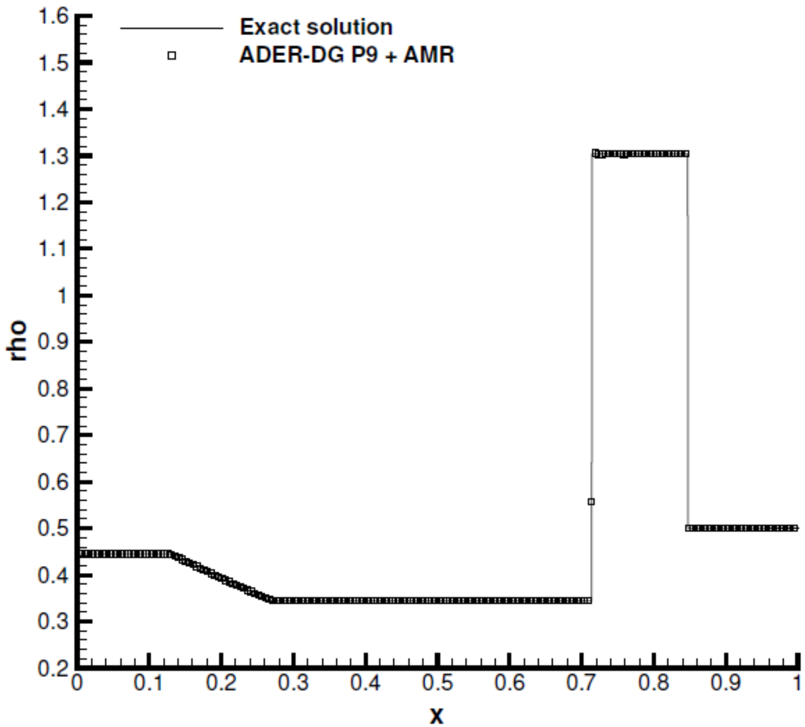
# Coupling of *a posteriori* subcell limiters for DG with AMR




ADER-DG (N=9) with AMR. Unlimited cells (blue) and limited cells (red)

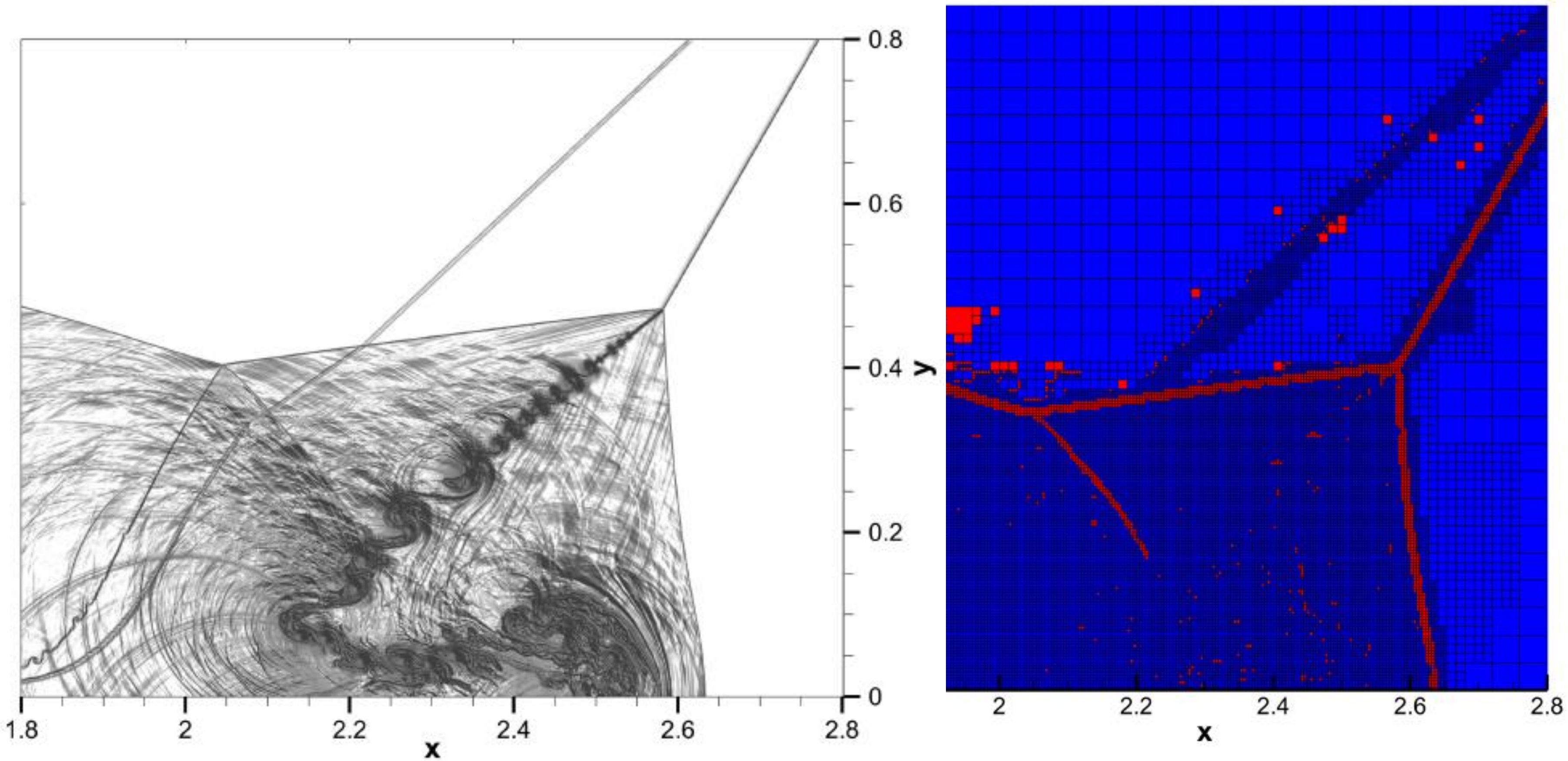
# Coupling of AMR with a posteriori subcell limiters for DG



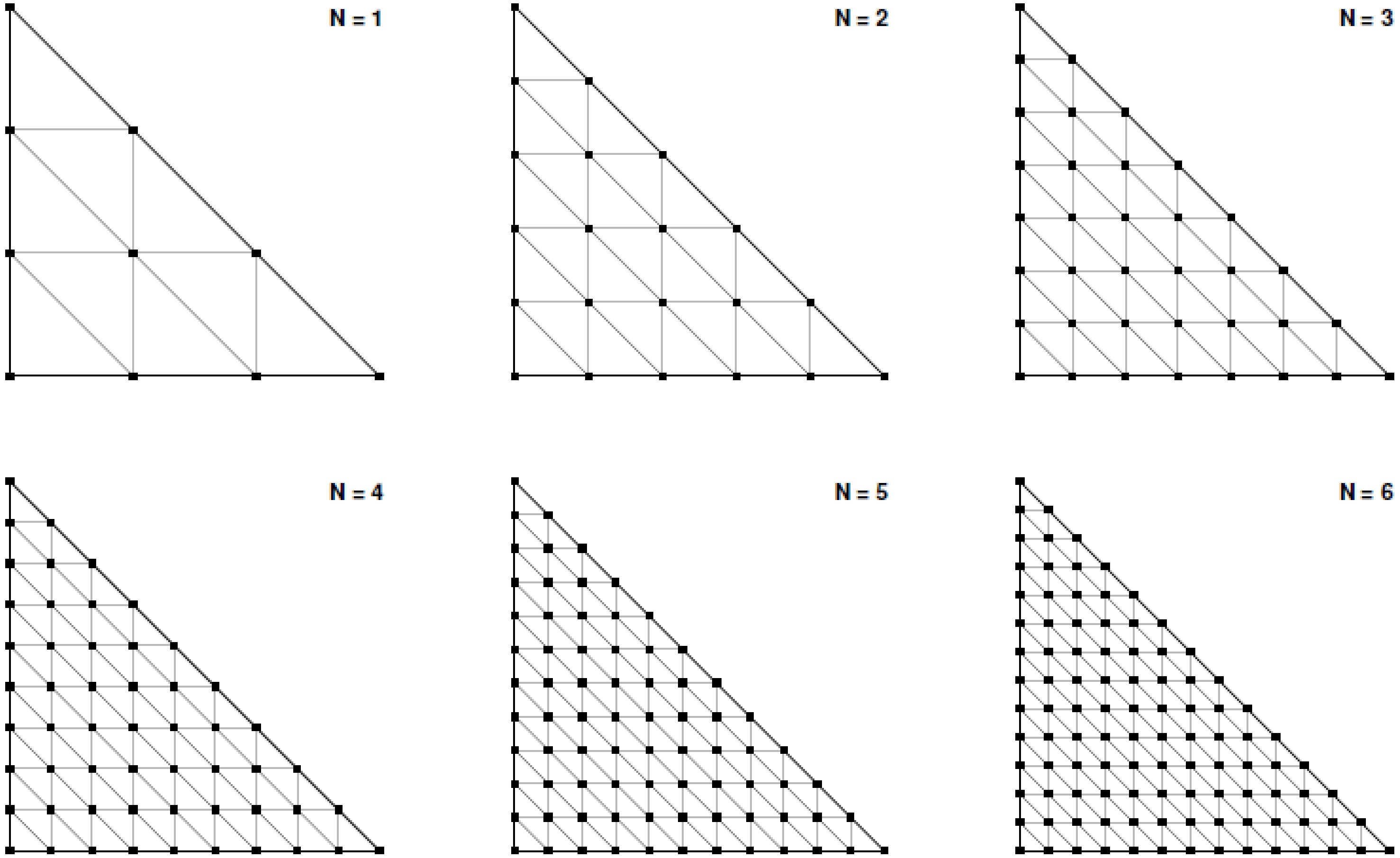

**ADER-DG (N=9) with a posteriori ADER-WENO subcell limiter and space-time adaptive mesh refinement (AMR) yields an <u>unprecedented resolution</u> of shocks and contact waves.**

# Coupling of AMR with a posteriori subcell limiters for DG



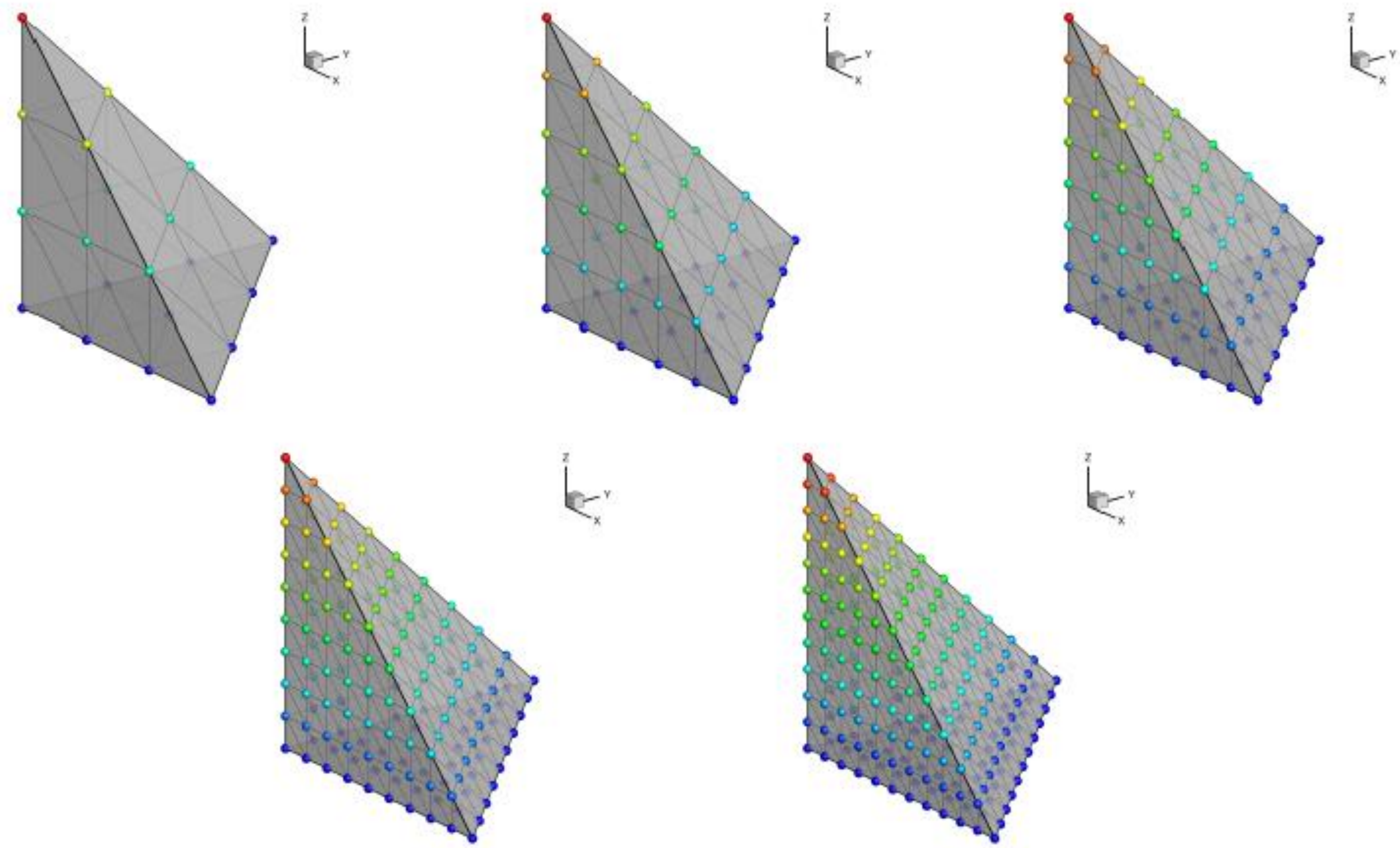

**Double Mach reflection problem using ADER-DG (N=9) with a posteriori ADER-WENO subcell limiter and space-time adaptive mesh refinement (AMR)**

# Natural Extension to Unstructured Meshes



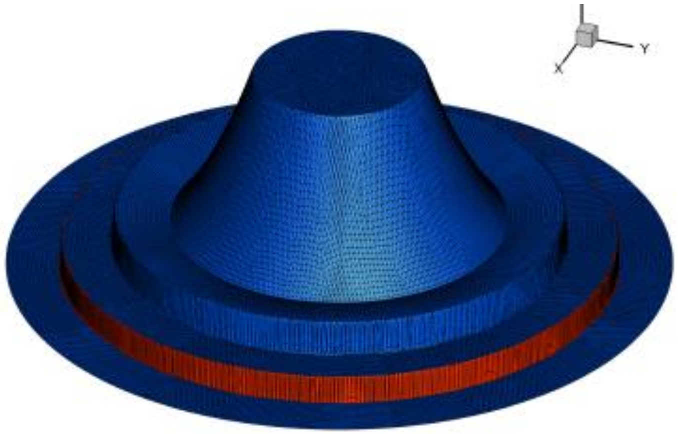

**Subgrid for N=1 to N=6 in 2D**

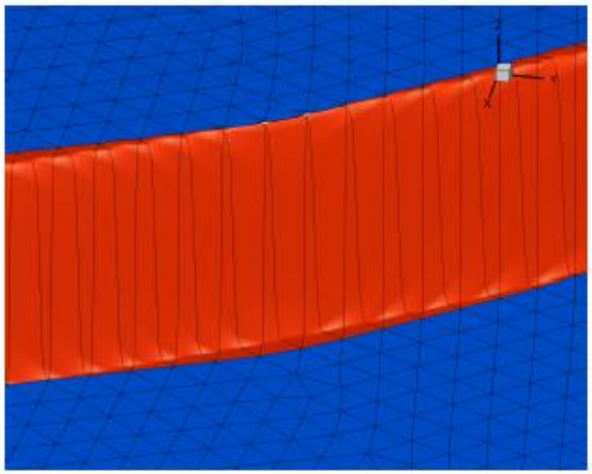
# Natural Extension to Unstructured Meshes

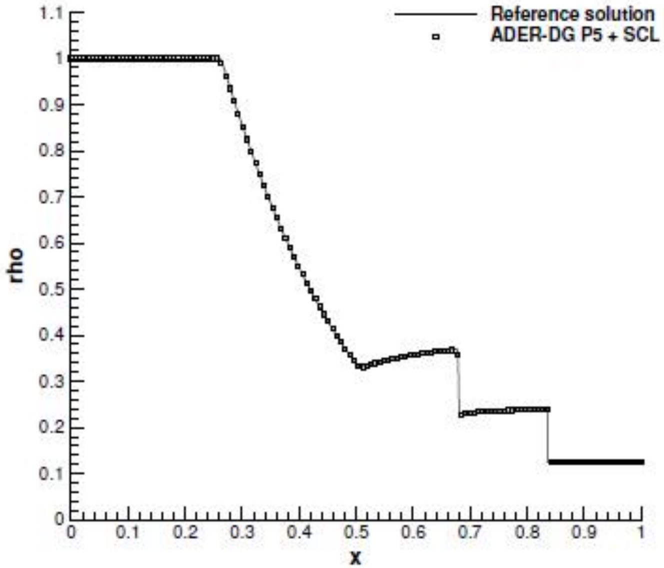


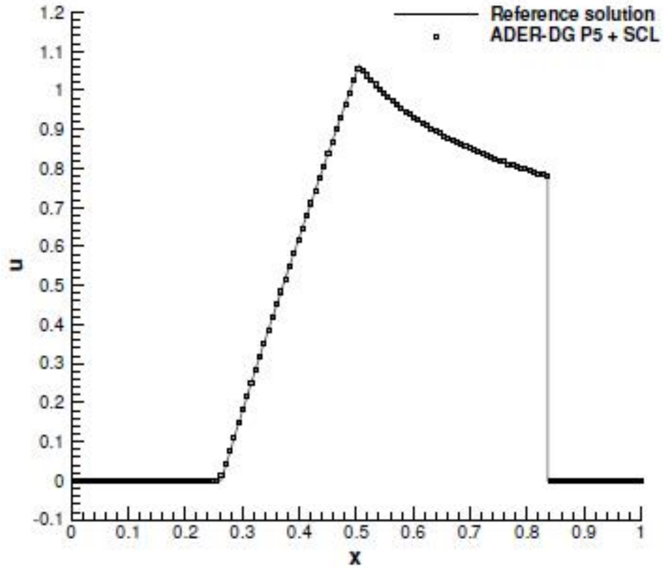
**Subgrid for N=1 to N=5 in 3D**

# Natural Extension to Unstructured Meshes






**Circular explosion problem in 2D (*N*=5)**

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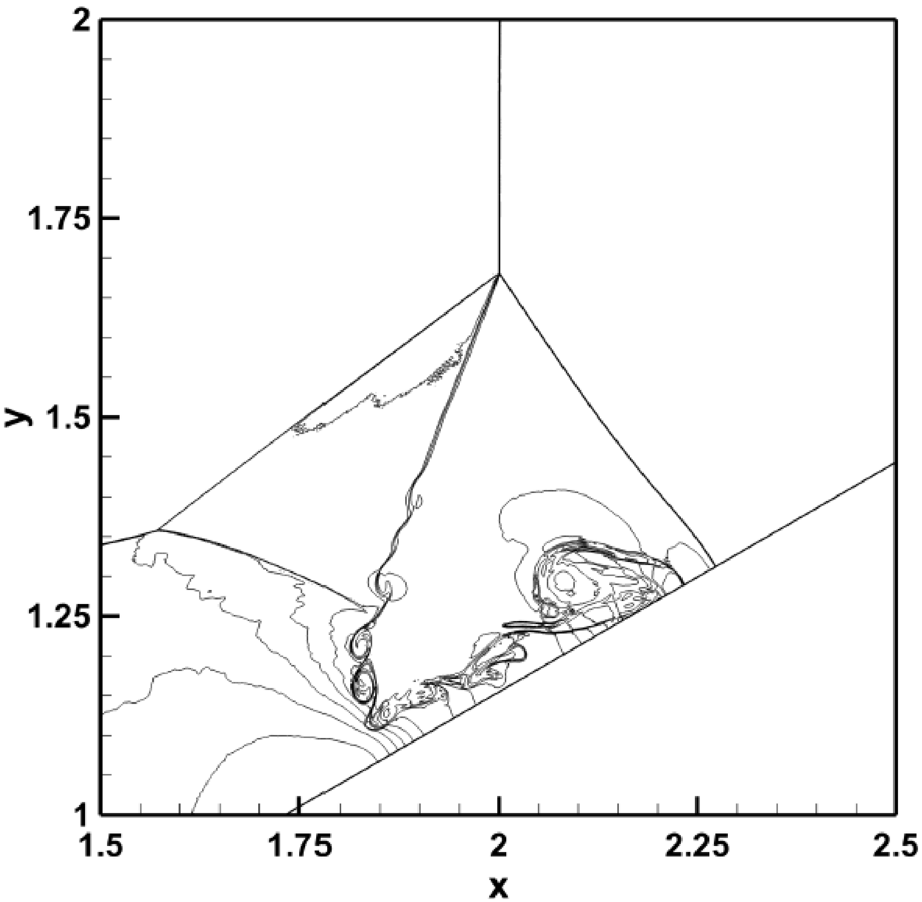


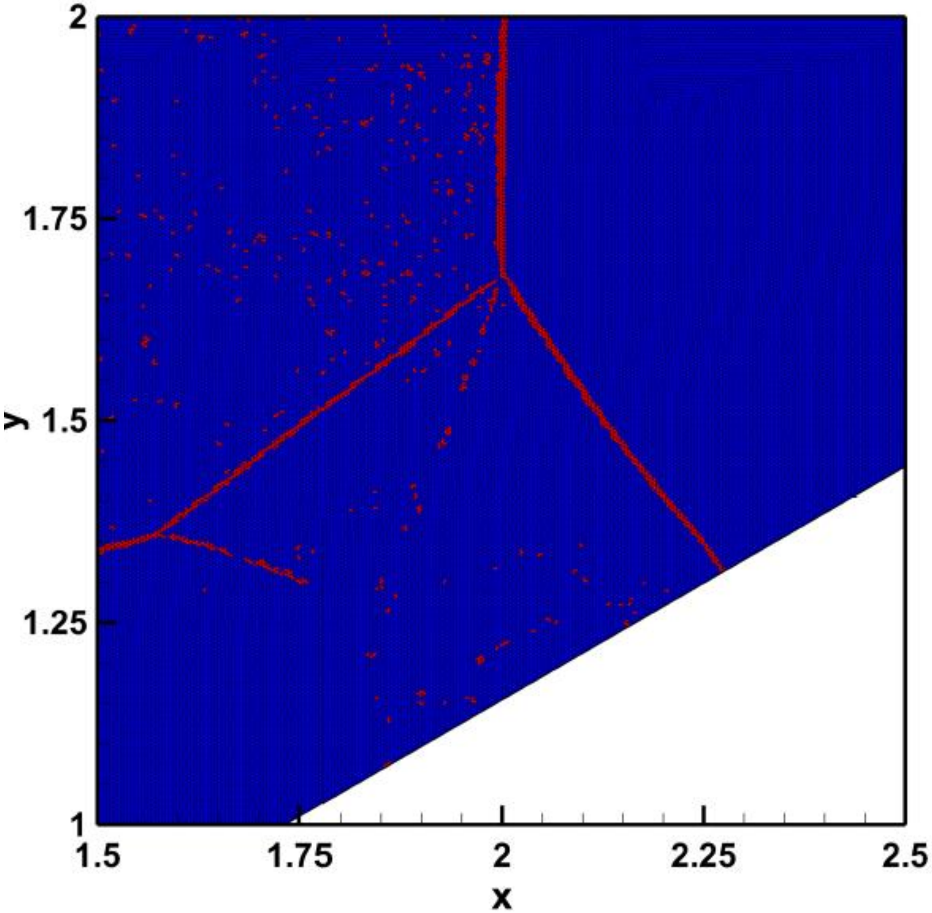


**Double Mach reflection problem in 2D (*N*=4)**

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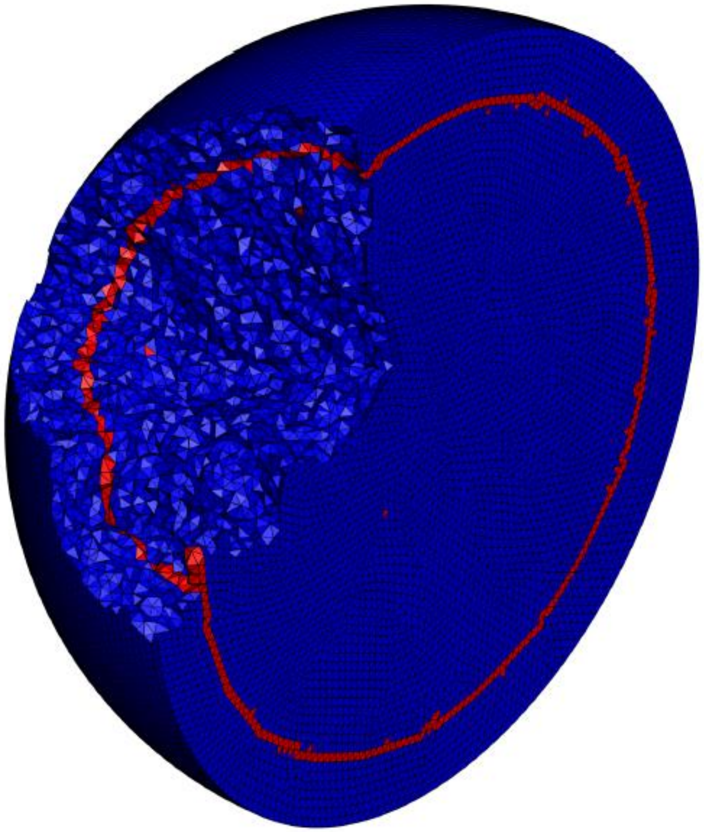
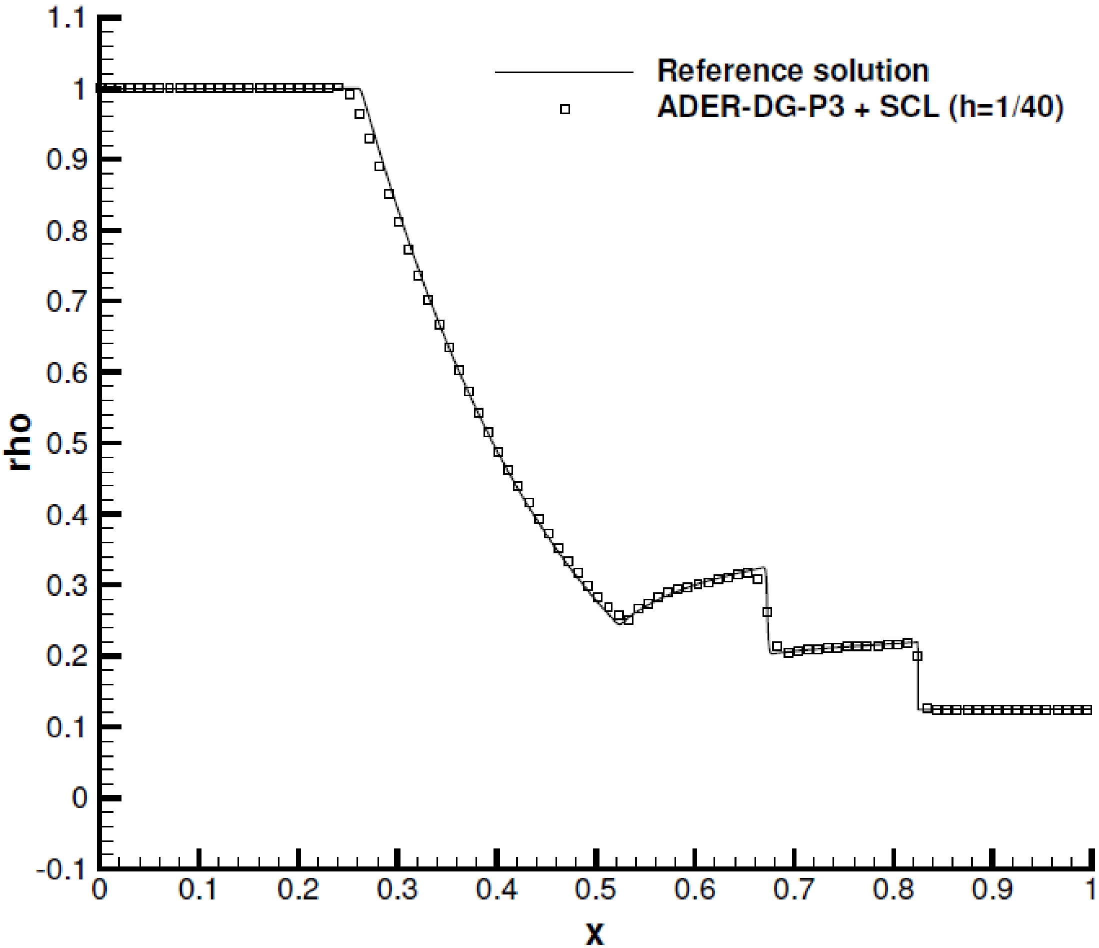
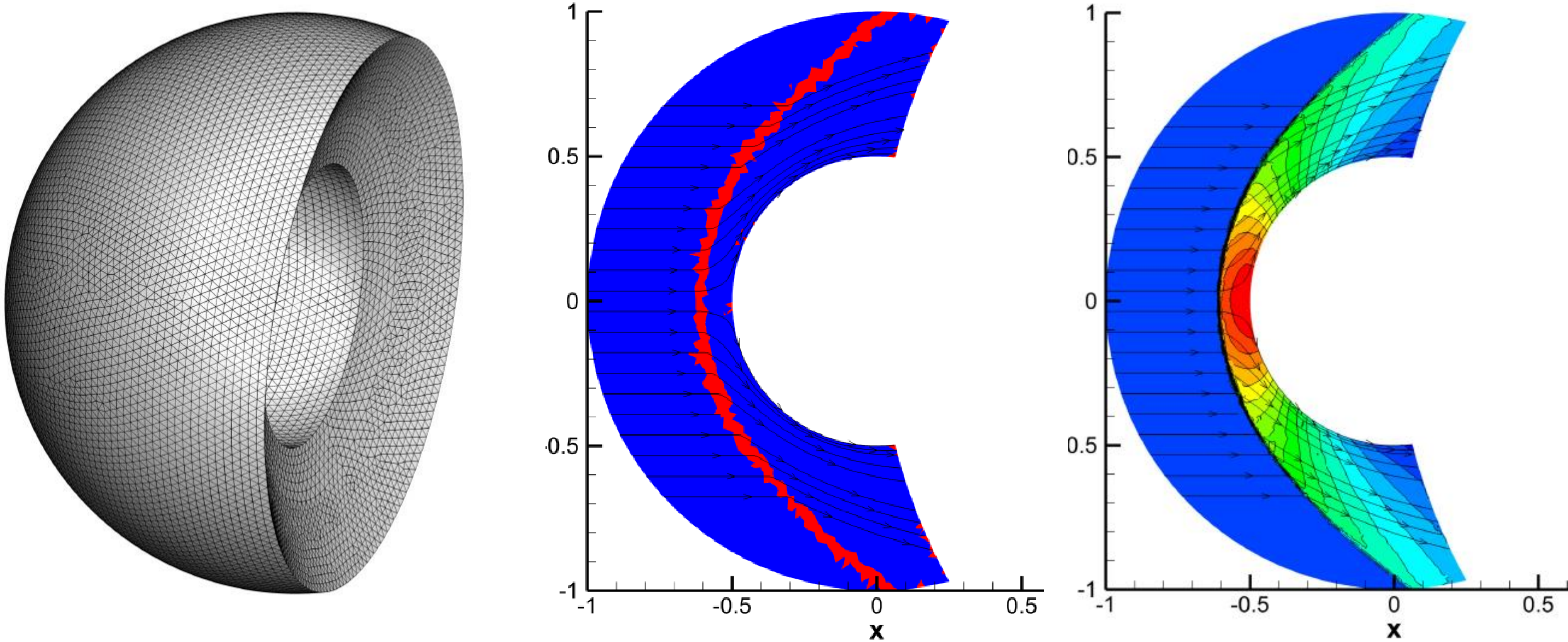





**Spherical explosion problem in 3D (*N*=3)**

# Natural Extension to Unstructured Meshes




**Mach 3 flow over a sphere (*N*=3)**

# Conclusions

- New, simple robust and accurate *a posteriori* subcell finite volume limiter for the discontinuous Galerkin finite element method
- High order fully discrete one-step ADER time discretization
- Available for uniform and space-time adaptive (AMR) Cartesian grids as well as for general triangular and tetrahedral unstructured meshes
- The *a posteriori* MOOD framework of Loubère, Clain and Diot has been found to be an ideal framework to devise a simple and robust limiter for DG schemes
- Why *a posteriori*: It is much simpler to **observe** (and cure) the occurrence of a troubled cell rather than to **predict** (and avoid) its occurrence from given data.
- Element-local **checkpointing** and solver **restarting** is even able to **cure** floating point errors (**NaN,** e.g. after division by zero)
- Future extension: **Lagrangian-type DG schemes** on unstructured ALE meshes