Fourth International Workshop
"Computational Experiment in AeroAcoustics"
September 21-24, 2016, Svetlogorsk, Russia
CEAA

Unsteady flux correction method for solving aeroacoustic problems on unstructured meshes

P. A. Bakhvalov, T. K. Kozubskaya

Schemes on unstructured meshes

Schemes for solving Euler equations

Very high (3rd and higher) order methods

- Discontinious Galerkin
- FV polynomial-based
- SD, SV, etc.

- Very good accuracy for smooth solutions
- Enforcing monotonicity leads to loss of accuracy
- Very expensive, especially for problems with discontinuities

2nd order Godunov-type methods

- cell-centered schemes
- vertex-centered (edge-based) schemes

- Cheap
- Easy to realize
- Just 2nd order

Schemes on unstructured meshes

Schemes for solving Euler equations

Very high (3rd and higher) order methods

- Discontinious Galerkin
- FV polynomial-based
- SD, SV, etc.

- Very good accuracy for smooth solutions
- Enforcing monotonicity leads to loss of accuracy
- Very expensive, especially for problems with discontinuities

- Constructing new limiters, etc.

2nd order Godunov-type methods

- cell-centered schemes
- vertex-centered (edge-based) schemes

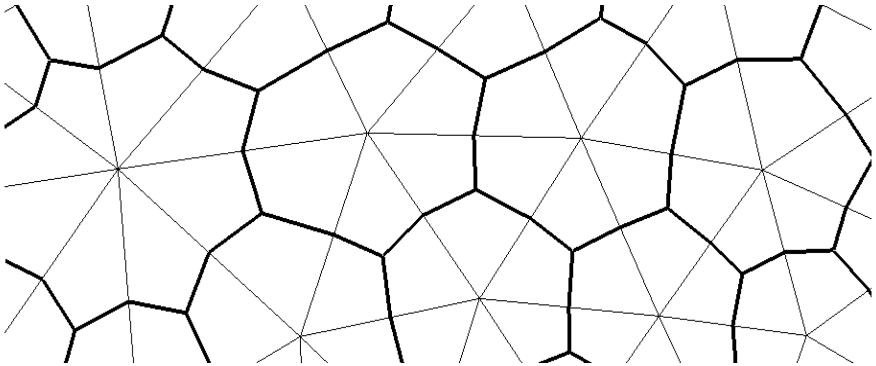
- Cheap
- Easy to realize
- Just 2nd order

- Increasing accuracy within 2nd order of accuracy

«Linear» scheme of T. Barth

- 1980th: idea of equipping P1-Galerkin method with artificial dissipation
- Roe [1] found a finite-volume interpretation of P1-Galerkin method
- Barth [2] used this result to construct a 2nd order scheme consisting of linear reconstruction of variables and Godunov-type Riemann solver
- Now «Linear» scheme **is an ancestor of most vertex-centered schemes**

$$[\nabla\cdot\mathbf{F}]|_i = \frac{1}{v_i}\sum_{j\in N(i)}\left(\frac{\mathbf{F}(Q_i)+\mathbf{F}(Q_j)}{2} - \frac{1}{2}\left(\frac{d\mathbf{F}}{dQ}\left(\overline{Q}_{ij}\right)\right)(Q_{ij}^{+}-Q_{ij}^{-})\right)\cdot\mathbf{n}_{ij}$$



Barycentric control volumes define V_i and $\mathbf{n}_{ij}$

[1] P. L. Roe, "Error estimates for cell-vertex solutions of the compressible Euler equations", ICASE report No. 87-6 (1987)

[2] T. J. Barth, "A 3-D upwind Euler solver for unstructured meshes", AIAA Paper No. 91-1548 (1991)

Increasing accuracy of 2nd order schemes

Two main approaches based on «Linear» scheme of T. Barth

Quasi-1D approach

- Keeps 1-exactness
- Uses quasi-1D reconstruction of variables
- TVD and WENO techniques for discontinuities

- Reduces to 5th order FD scheme for uniform grid-like meshes
- Cheap
- 2nd order for unstructured meshes but better accuracy than «Linear» scheme

Flux correction approach

- Makes 2-exactness (for smooth solutions)
- Uses spectral elements for gradient approximations
- TVD and SLIP limiters for discontinuities

- 3rd order for steady problems
- 2nd order for unsteady problems with no accuracy improvement or complex dQ/dt term approximation with loss of conservation

Increasing accuracy of 2nd order schemes

Two main approaches based on «Linear» scheme of T. Barth

Edge-based approach

- Keep 1-exactness
- Use quasi-1D reconstruction of variables
- TVD and WENO approaches for discontinuities

UFC

- Keep 2-exactness only for steady problems (for smooth solutions) and 1-exactness for unsteady
- Use spectral elements for gradient approximations
- TVD and SLIP limiters

Flux correction approach

- Keep 2-exactness (for smooth solutions)
- Use spectral elements for gradient approximations
- TVD and SLIP limiters for discontinuities
- 3rd order for steady problems
- 2nd order for unsteady problems with no accuracy improvement or complex dQ/dt term approximation with loss of conservation

- 3rd order for steady problems
- 3rd order for uniform grid-like meshes
- Still cheap
- 2nd order with less numerical error for unsteady problems

Desirable properties of Edge-based schemes

- 1 - conservation
- 2 - exactness on linear functions (1-exactness)
- 3 – high (3^{rd}-5^{th}) order of accuracy on uniform structured meshes
- 4 - 3^{rd} order of accuracy for steady problems on arbitrary meshes
- 5 - 3^{rd} order of accuracy for unsteady problems on arbitrary meshes
- 6 - significantly cheaper than very high order scheme

	1	**2**	**3**	**4**	**5**	**6**
Linear scheme	+	+	-	-	-	+
EBR, SEBR, MEV, NLV6, LV6, ..	+	+	+	-	-	+
UFC	+	+	**+**	**+**	-	+
steady FC	+	+	-	+	-	+
FC modifications of Nishikawa, Pincock	-	+	+	+	+	-

Edge-based vertex-centered schemes

$$\frac{\partial \mathbf{Q}}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{Q}) = 0$$

Euler Eqs

$$\mathbf{Q} = \left(\rho, \rho\mathbf{u}, E\right)^T$$

$$\mathcal{F} = \begin{pmatrix} \rho\mathbf{u} \\ \rho\mathbf{u}\mathbf{u} + p\mathbf{I} \\ \left(E + p\right)\mathbf{u} \end{pmatrix}$$

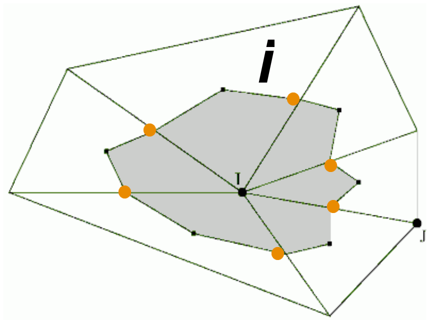
Linearised Euler Eqs

$$\mathbf{Q} = \left(\rho', \mathbf{u}', p'/(\gamma-1)\right)^T$$

$$\mathcal{F} = \begin{pmatrix} \mathbf{u}' \\ p'\mathbf{I} \\ \mathbf{u}'/(\gamma-1) \end{pmatrix}$$

$$\left(\frac{d\mathbf{Q}}{dt}\right)_i = -\frac{1}{v_i} \sum_{j \in N_1(\mathrm{i})} \mathbf{h}_{ij}, \quad \mathbf{h}_{ij} = -\mathbf{h}_{ji}$$

$$\mathbf{h}_{ij} = \mathcal{F}_{ij} \cdot \mathbf{n}_{ij} \qquad \mathbf{n}_{ij} = \int_{\partial C_{ij}} \mathbf{n}\, ds$$



$\mathcal{F}_{ij}$ evaluated at edge *ij* midpoint

Flux Correction (FC) Scheme

$$\frac{\partial u}{\partial t} + \mathbf{a} \cdot \nabla u = 0$$

$$\boxed{\frac{\mathrm{d}u_i}{\mathrm{d}t} + \left[\mathbf{a} \cdot \nabla u\right]_i^{FC} = 0}$$

$$\left[\nabla u(x)\right]_i^{FC} = \frac{1}{v_i} \sum_{j \in N_1(i)} \mathbf{n}_{ij} u_{ij}$$

$$u_{ij} = \begin{cases} u_{ij}^{(i)}, \ \mathbf{a}_{ij} \cdot \mathbf{n}_{ij} \geq 0 \\ u_{ij}^{(j)}, \ \mathbf{a}_{ij} \cdot \mathbf{n}_{ij} < 0 \end{cases}$$

$$u_{ij}^{(i)} = u_i + \frac{1}{2}(\mathbf{r}_j - \mathbf{r}_i) \cdot \nabla u(\mathbf{r}_i)$$

$$u_{ij}^{(j)} = u_j + \frac{1}{2}(\mathbf{r}_i - \mathbf{r}_j) \cdot \nabla u(\mathbf{r}_j)$$

$\nabla u(\mathbf{r}_i), \nabla u(\mathbf{r}_j)$ are calculated accurately enough

Katz A., and Sankaran V., “An Efficient Correction Method to Obtain a Formally Third-Order Accurate Flow Solver for Node-Centered Unstructured Grids,” *Journal of Scientific Computing*, Vol. 51, 2012

Construction of the UFC scheme

Original steady FC scheme: $\frac{\mathrm{d}u}{\mathrm{d}t}+[\mathbf{a}\cdot\nabla u]^{FC}=0$ – 2nd order of accuracy

UFC scheme on TS meshes: $\frac{\mathrm{d}u}{\mathrm{d}t}+\hat{U}\,[\mathbf{a}\cdot\nabla u]^{FC}=0$ – 3rd order of accuracy

$$\hat{u}_{ik}=\begin{cases} C/12, & k\in N_1(i) \\ 1-C|N_1(i)|/12, & k=i \\ 0 & otherwise \end{cases} \qquad C=\begin{cases} 1, & 1\mathrm{D} \\ 1/2, & 2\mathrm{D} \\ 1/4, & 3\mathrm{D} \end{cases}$$

UFC scheme on arbitrary simplicial mesh:

1. Scheme must be conservative
2. Scheme must be 1-exact
3. U must coincide with $\hat{U}$ on TS-meshes

$$\boxed{\frac{\mathrm{d}u}{\mathrm{d}t}+U\,[\mathbf{a}\cdot\nabla u]^{FC}=0}$$

$$u_{ik}=\begin{cases} \frac{C}{12}\frac{v_i+v_k}{2v_i}, & k\in N_1(i) \\ 1-\frac{C}{12}\sum\limits_{k\in N_1(i)}\frac{v_i+v_k}{2v_i}, & k=i \\ 0 & otherwise \end{cases}$$

2D and 3D numerical results

Linearized Euler equations,   Gaussian pulse evolution

$$\bar{\rho} = 1 \qquad \bar{\mathbf{u}} = 0 \qquad \bar{p} = 1/\gamma$$

$$\rho'\big|_{t=0} = p'\big|_{t=0} = A\exp\left(-\ln 2\left(\frac{\mathbf{r}}{b}\right)^2\right) \qquad A = \begin{cases} 1, & 2\text{D} \\ 1/2, & 3\text{D} \end{cases} \qquad b = 6$$

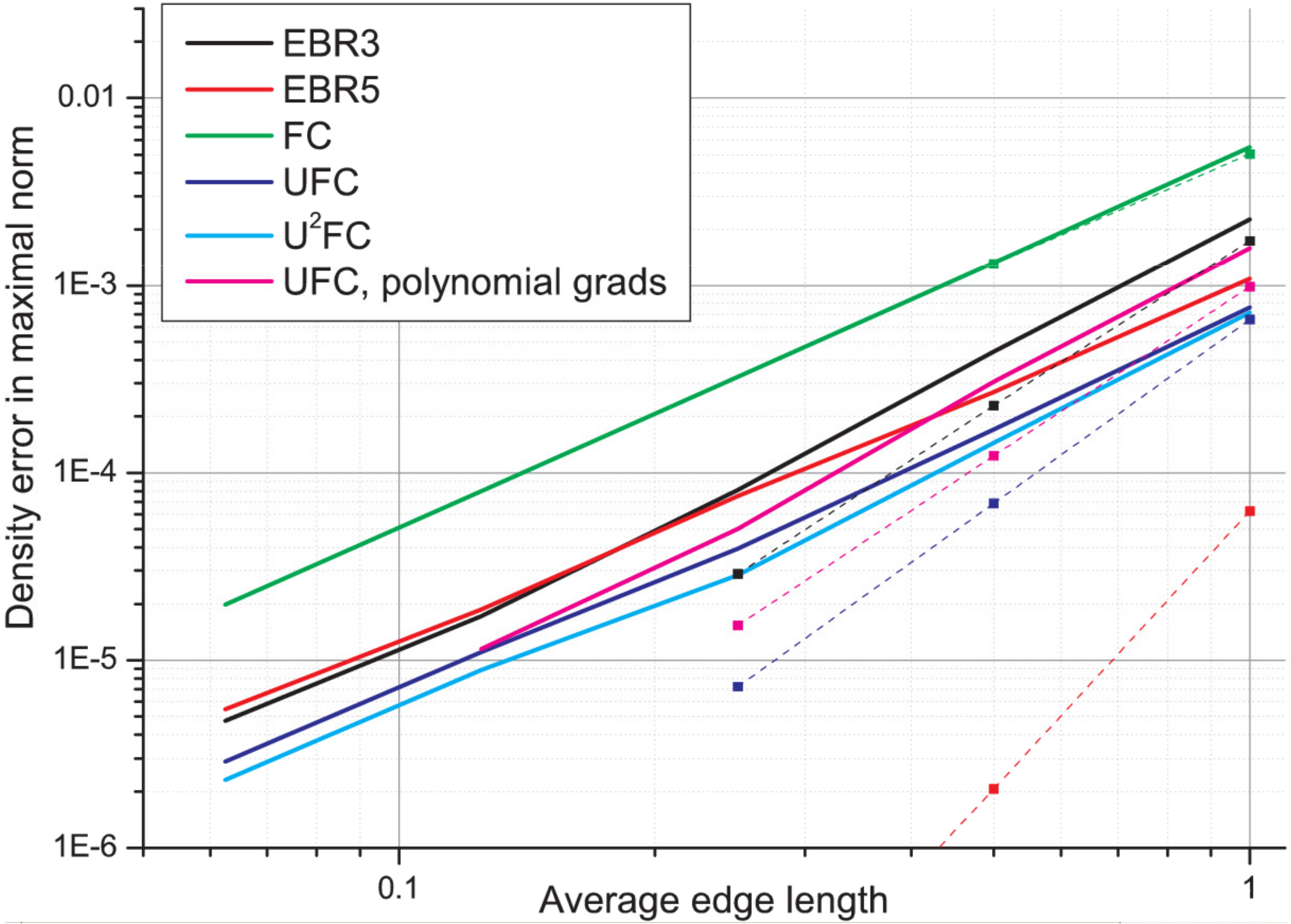
$$T = 40$$

Scheme		EBR3	EBR5	Steady FC	UFC
Maximal norm	h=1 h=0.5 h=0.25 num. order	1.73289e-003 2.28420e-004 2.89376e-005 2.98	6.24980e-005 2.05613e-006 7.16193e-008 4.84	5.03122e-003 1.30860e-003 3.28580e-004 2.00	6.57931e-004 6.86734e-005 7.22010e-006 3.25
Integral norm	h=1 h=0.5 h=0.25 num. order	2.77561e-004 3.55518e-005 4.45939e-006 3.00	9.39937e-006 3.01253e-007 1.13177e-008 4.73	7.23611e-004 1.85495e-004 4.67779e-005 1.99	5.43120e-005 5.04358e-006 6.59735e-007 2.93

Triangular TS-meshes

Maximal norm	h=1 h=0.5 h=0.25 h=0.125 h=0.0625 num. order	2.2576e-003 4.42802e-4 8.08705e-5 1.72135e-5 4.76772e-6 1.85	1.08382e-003 2.69707e-4 7.52786e-5 1.85921e-5 5.46002e-6 1.76	5.47836e-003 1.32e-3 3.2483e-4 7.98531e-5 1.97773e-5 2.01	7.63247e-4 1.70302e-4 3.94256e-5 1.10177e-5 2.87777e-6 1.94
Integral norm	h=1 h=0.5 h=0.25 h=0.125 h=0.0625 num. order	2.78876e-004 3.10223e-005 4.33573e-06 6.89182e-07 1.39072e-07 2.31	7.10360e-005 1.24319e-005 2.91023e-06 7.42516e-07 1.85712e-07 2.00	7.50898e-004 1.79015e-004 4.51444e-05 1.13290e-05 2.81990e-06 2.01	6.00207e-005 6.97642e-006 1.53294e-06 3.95812e-07 9.91140e-08 2.00

Quasi-uniform unstructured triangular meshes



Solution convergence on triangular meshes:
unstructured mesh versus TS-mesh (of the same color in dash)

Scheme		EBR3	EBR5	Steady FC	UFC
Maximal norm	h=1	6.71e-3	7.41e-4	1.15e-2	4.10e-3
	h=1/2	9.71e-4	2.63e-5	2.75e-3	5.14e-4
	num. order	2.79	4.84	2.06	3.00
Integral norm	h=1	1.66e-3	1.59e-4	3.12e-3	9.78e-4
	h=1/2	2.31e-4	7.71e-6	8.03e-4	1.21e-4
	num. order	2.85	4.88	1.96	3.01
	theor. order	3	5	2	3

Tetrahedral TS-meshes

Scheme		EBR3	EBR5	Steady FC	UFC
Maximal norm	h=1	4.79e-3	1.82e-3	1.39e-2	2.82e-3
	h=1/2	6.54e-4	4.36e-4	3.21e-3	3.51e-4
	h=1/4	1.04e-4	1.09e-4	7.68e-4	5.99e-5
	num. order	2.65	2.41	2.00	2.55
Integral norm	h=1	7.23e-4	2.50e-4	2.35e-3	4.45e-4
	h=1/2	9.70e-5	5.15e-5	6.02e-4	5.70e-5
	h=1/4	1.37e-5	1.24e-5	1.51e-4	8.22e-6
	num. order	2.94	2.03	2.01	2.79
	theor. order	2	2	2	2

Quasi-uniform unstructured tetrahedral meshes

Conclusion

We have presented the **Unsteady flux correction (UFC)** method.

At this point we have developed the scheme for smooth solution only.

For linear case the UFC scheme is:

- conservative;
- of second order on arbitraty meshes;
- of third order on uniform grid-like meshes;
- of third order on steady problems;
- as cheap as the original steady FC method.

FC scheme is easy to realize and need no geometry preprocessing
so it is well suitable for solving problems on deforming meshes.