WENO-EBR Scheme for Solving Aerodynamics and Aeroacoustics Problems on Unstructured Meshes

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Introduction

In aeroacoustics problems
near field flow region is considered as a distributed acoustic source
which radiates sound usually evaluated in far fields with the help of integral methods (FWH)

Accuracy of far field acoustics strongly depends on accuracy of near field simulation

We develop efficient low**ER**-cost and high**ER-**accuracy schemes
for simulating near flow fields (including turbulent flows)
on unstructured meshes
aiming at **real aeroacoustics problems**.

Our method is Vertex-centered **EBR (Edge-Based Reconstruction) schemes**
based on quasi-1D edge-oriented reconstruction of variables.

The talk represents the further development of **WENO-EBR schemes**
firstly presented at CEAA2012

1. Dervieux, A. and Debiez, C. Mixed element volume MUSCL methods with weak visco-sity for steady and unsteady flow calculation. *Computer and Fluids* 29: 89-118, 1999
2. Abalakin, I., A. Dervieux and T. Kozubskaya. High Accuracy Finite Volume Method for Solving Nonlinear Aeroacoustics Problems on Unstructured Meshes. *Chinese Journal of Aeroanautics* 19: 97-104, 2006
3. Ilya Abalakin, Pavel Bakhvalov and Tatiana Kozubskaya. Edge-based reconstruction schemes for prediction of near field flow region in complex aeroacoustic problems, *International Journal of Aeroacoustics*, Vol.13, N 3&4, 2014, p. 207-234.
4. Ilya Abalakin, Pavel Bakhvalov, Tatiana Kozubskaya. Edge-Based Methods in CAA. – In *“Accurate and Efficient Aeroacoustic Prediction Approaches for Airframe Noise”,* Lecture Series 2013-03, Ed. by C.Schram, R.Denos, E.Lecomte, von Karman Institute for Fluid Dynamics, (2013) (ISBN-13 978-2-87516-048-5).

About EBR Schemes Briefly

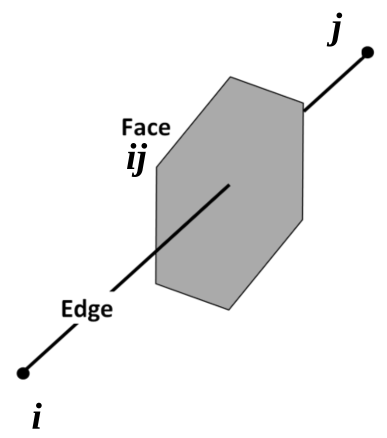
1. **Method of lines** for solving the system of ODEs

2. **Vertex-centered formulation** — a mesh node i is a center of the cell of volume V_i
   All the geometry (node neighbors, cells, faces, normals,…) is built once
   at the pre-processing stage.

3. **Conservation laws** result from the general formulation:

$$\left[\frac{\partial \mathbf{Q}}{\partial t}\right]_i + \frac{1}{V_i}\sum_{j\in N(i)} \mathcal{F}_{ij} S_{ij} = 0$$

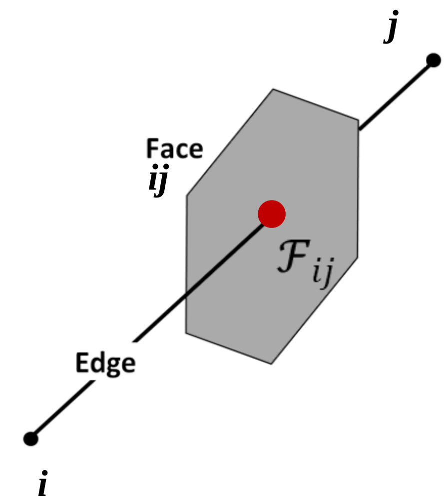
$N(i)$ — a set of cell faces for node i
(or edges that are incident to node i)
$\mathcal{F}_{ij}$ — flux through the cell face ij of square S_{ij}
(projected on the normal)



4. **Edge-wise implementation**:
the main loop is on edges with accumulating the necessary integral sums.

How to determine the fluxes $\mathcal{F}_{ij}$?

Let us determine the flux through each cell face
at a single point = the middle of corresponding edge



$$\mathcal{F}_{ij} = h\left(\mathbf{Q}_{ij}^{L}, \mathbf{Q}_{ij}^{R}\right) = \frac{\mathbf{F}\left(\mathbf{Q}_{ij}^{L}\right) + \mathbf{F}\left(\mathbf{Q}_{ij}^{R}\right)}{2} - \frac{\delta}{2}\left|\mathbf{A}_{ij}\right|\left(\mathbf{Q}_{ij}^{R} - \mathbf{Q}_{ij}^{L}\right) \quad \text{Roe}$$

$$\mathcal{F}_{ij} = h\left(\mathbf{F}_{ij}^{L}, \mathbf{F}_{ij}^{R}, \mathbf{Q}_{i}, \mathbf{Q}_{j}\right) = \frac{\mathbf{F}_{ij}^{L} + \mathbf{F}_{ij}^{R}}{2} - \frac{\delta}{2}\operatorname{sign}\mathbf{A}_{ij}\left(\mathbf{Q}_{i}, \mathbf{Q}_{j}\right)\left(\mathbf{F}_{ij}^{R} - \mathbf{F}_{ij}^{L}\right) \quad \text{Huang}$$

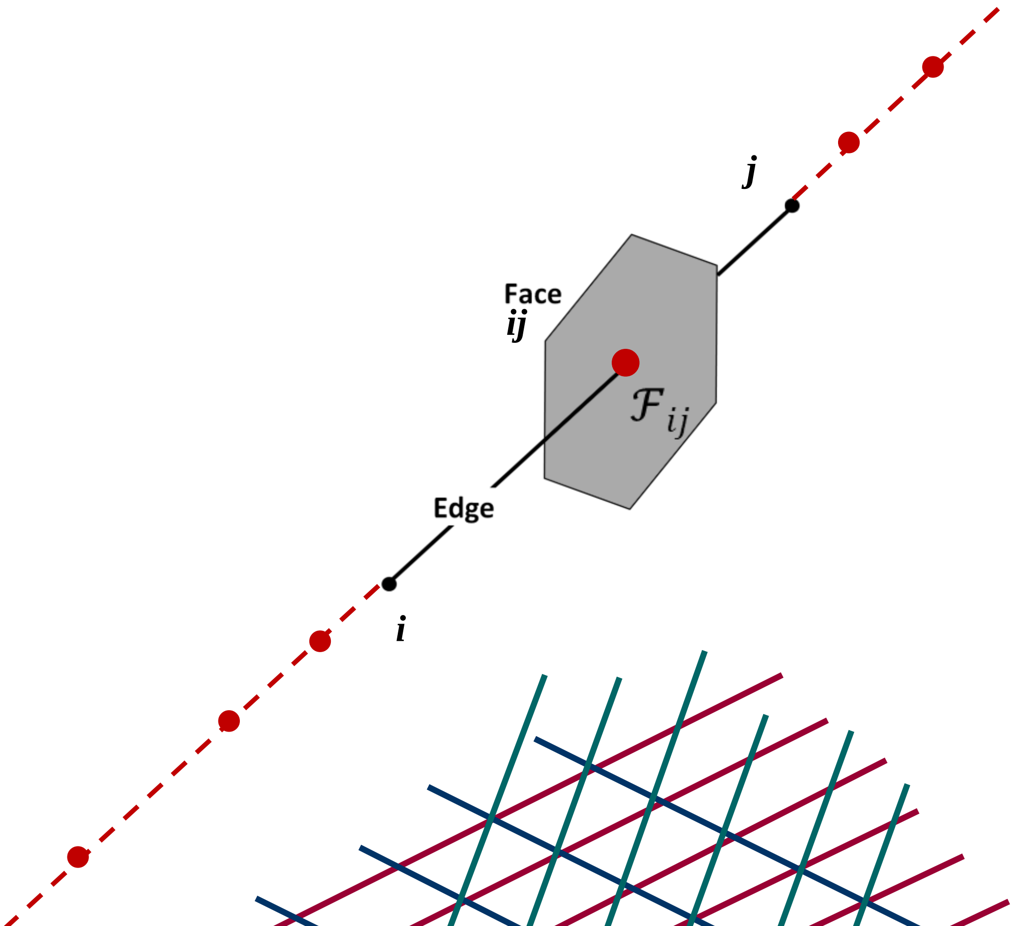
$$\mathcal{F}_{ij} = h\left(\mathbf{F}_{ij}^{L}, \mathbf{F}_{ij}^{R}, \mathbf{Q}_{i}, \mathbf{Q}_{j}, \mathbf{Q}_{ij}^{L}, \mathbf{Q}_{ij}^{R}\right) = \frac{\mathbf{F}_{ij}^{L} + \mathbf{F}_{ij}^{R}}{2} - \frac{\delta}{2}\left|\mathbf{A}\left(\mathbf{Q}_{i}, \mathbf{Q}_{j}\right)\right|_{ij}\left(\mathbf{Q}_{ij}^{R} - \mathbf{Q}_{ij}^{L}\right) \quad \text{Hybrid}$$

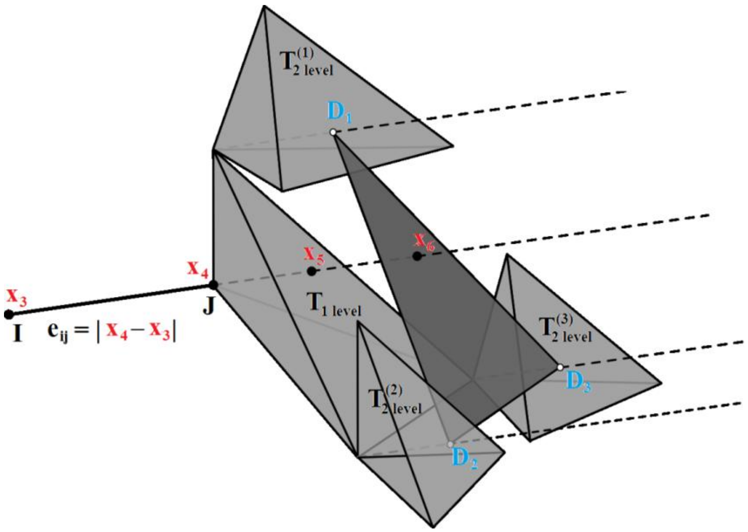
$$\mathcal{F}_{ij} = \left(\mathbf{F}^{+}\right)_{ij}^{L} + \left(\mathbf{F}^{-}\right)_{ij}^{R}, \quad \mathbf{F} = \mathbf{F}^{+} + \mathbf{F}^{-} \qquad \text{Flux splitting methods}$$

$()^{L/R}$ — Left/Right Reconstruction of mesh function on cell interfaces

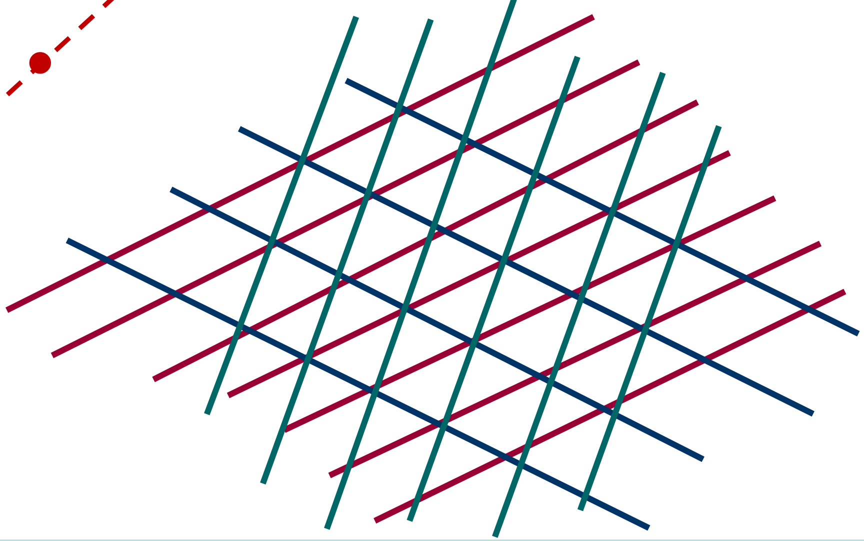
$$\mathbf{F} = \mathbf{F}_{x} n_{x} + \mathbf{F}_{y} n_{y} + \mathbf{F}_{z} n_{z}, \quad \mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{Q}} = \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^{-1}$$

Quasi-1D Reconstruction - Stencil



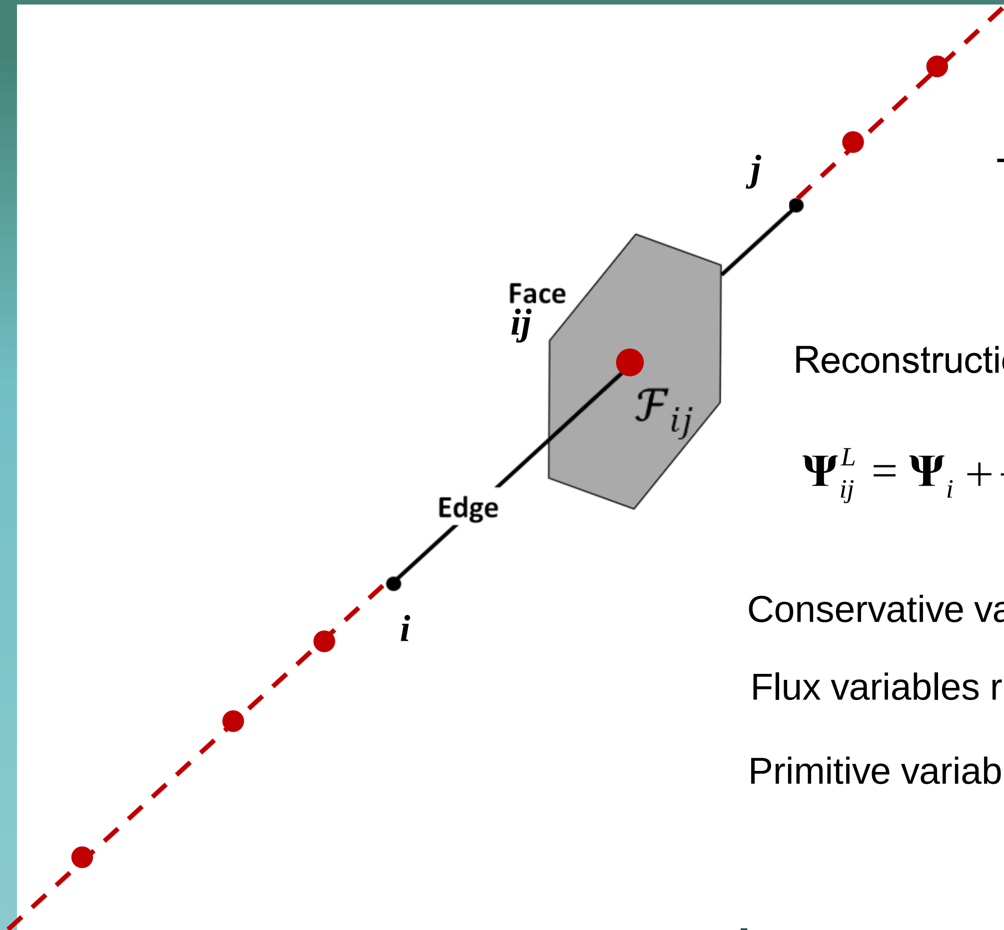


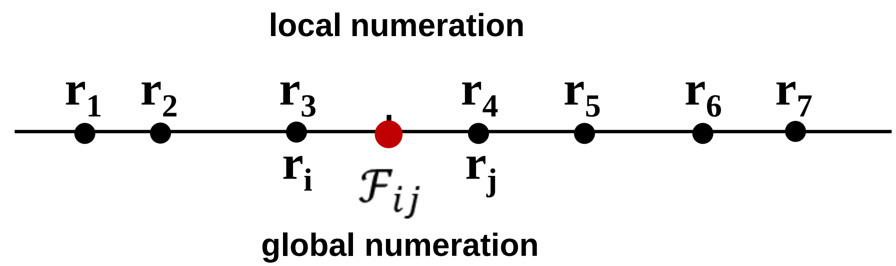
One way of finding the stencil points



EBR schemes reach the highest
order at **TS-meshes**
(translationally-symmetric)

Quasi-1D Reconstruction





Reconstruction variables (variables in vertices + "slopes")

$$\mathbf{\Psi}_{ij}^{L} = \mathbf{\Psi}_{i} + \frac{1}{2}\sum_{k=1}^{m}\beta_{ij,k}^{L}\Delta\mathbf{\Psi}_{k+1/2}, \quad \mathbf{\Psi}_{ij}^{R} = \mathbf{\Psi}_{j} - \frac{1}{2}\sum_{k=1}^{m}\beta_{ij,k}^{R}\Delta\mathbf{\Psi}_{k+1/2}$$

Conservative variables reconstruction $\mathbf{\Psi}_{ij}^{L/R} = \mathbf{Q}_{ij}^{L/R}$

Flux variables reconstruction $\mathbf{\Psi}_{ij}^{L/R} = \mathbf{F}_{ij}^{L/R}$

Primitive variables reconstruction $\mathbf{\Psi}_{ij}^{L/R} = \mathbf{Q}\left(\mathbf{U}_{ij}^{L/R}\right)$

EBR5 scheme (m=4)

$$\beta_{ij,1}^{L} = -\frac{1}{15}\frac{\Delta r_{7/2}}{\Delta r_{3/2}}, \quad \beta_{ij,2}^{L} = \frac{11}{30}\frac{\Delta r_{7/2}}{\Delta r_{5/2}}, \quad \beta_{ij,3}^{L} = \frac{4}{5}\frac{\Delta r_{7/2}}{\Delta r_{7/2}}, \quad \beta_{ij,4}^{L} = -\frac{1}{10}\frac{\Delta r_{7/2}}{\Delta r_{9/2}}$$

$$\beta_{ij,1}^{R} = -\frac{1}{10}\frac{\Delta r_{7/2}}{\Delta r_{5/2}}, \quad \beta_{ij,2}^{R} = \frac{4}{5}\frac{\Delta r_{7/2}}{\Delta r_{7/2}}, \quad \beta_{ij,3}^{R} = \frac{11}{30}\frac{\Delta r_{7/2}}{\Delta r_{9/2}}, \quad \beta_{ij,4}^{R} = -\frac{1}{15}\frac{\Delta r_{7/2}}{\Delta r_{11/2}}$$

EBR3 scheme (m=2)

$$\beta_{ij,1}^{L} = \frac{1}{3}\frac{\Delta r_{7/2}}{\Delta r_{5/2}}, \quad \beta_{ij,2}^{L} = \frac{2}{3}\frac{\Delta r_{7/2}}{\Delta r_{7/2}}$$

$$\beta_{ij,1}^{R} = \frac{2}{3}\frac{\Delta r_{7/2}}{\Delta r_{7/2}}, \quad \beta_{ij,2}^{R} = \frac{1}{3}\frac{\Delta r_{7/2}}{\Delta r_{9/2}}$$

Quasi-1D Reconstruction

CH-WENO-EBR scheme

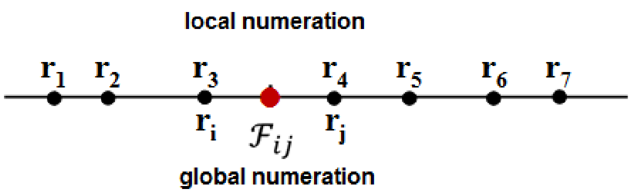
(for ”*L*” case)

1. Three 2nd order (m=2) Reconstructions of characteristic variables (different slopes) $\mathcal{L}_{ij} = \left[\mathbf{S}^{-1}\mathbf{Q}\right]_{ij}^{\text{Reconstructed}}$

$$\mathcal{L}_{ij}^{L(1)} = \mathbf{S}_{ij}^{-1}\mathbf{Q}_i + \mathbf{S}_{ij}^{-1}\frac{\Delta r_{7/2}}{2}\left(-\frac{1}{3}\frac{\Delta\mathbf{Q}_{3/2}}{\Delta r_{3/2}} + \frac{5}{3}\frac{\Delta\mathbf{Q}_{5/2}}{\Delta r_{5/2}}\right) = \mathbf{S}_{ij}^{-1}\mathbf{Q}_i + SlopeL(1)$$

$$\mathcal{L}_{ij}^{L(2)} = \mathbf{S}_{ij}^{-1}\mathbf{Q}_i + \mathbf{S}_{ij}^{-1}\frac{\Delta r_{7/2}}{2}\left(\frac{1}{3}\frac{\Delta\mathbf{Q}_{5/2}}{\Delta r_{5/2}} + \frac{2}{3}\frac{\Delta\mathbf{Q}_{7/2}}{\Delta r_{7/2}}\right) = \mathbf{S}_{ij}^{-1}\mathbf{Q}_i + SlopeL(2)$$

$$\mathcal{L}_{ij}^{L(3)} = \mathbf{S}_{ij}^{-1}\mathbf{Q}_i + \mathbf{S}_{ij}^{-1}\frac{\Delta r_{7/2}}{2}\left(\frac{4}{3}\frac{\Delta\mathbf{Q}_{7/2}}{\Delta r_{7/2}} - \frac{1}{3}\frac{\Delta\mathbf{Q}_{9/2}}{\Delta r_{9/2}}\right) = \mathbf{S}_{ij}^{-1}\mathbf{Q}_i + SlopeL(3)$$



$$\mathbf{S}_{ij}^{-1} = \mathbf{S}_{ij}^{-1}\left(\mathbf{Q}_i, \mathbf{Q}_j\right)$$

2. Weighted combination of the reconstructions

$$\mathcal{L}_{ij}^{L} = \sum_{k=1}^{3}\omega_k^L \mathcal{L}_{ij}^{L(k)}, \quad \omega_k^L = \frac{\sigma_k^L}{\sigma_1^L + \sigma_2^L + \sigma_3^L} \quad \sigma_k^L = \frac{\Omega^{(k)}}{\left(10^{-10} + IS_k^L\right)^2} \quad \Omega^{(1)} = \frac{1}{10} \quad \Omega^{(2)} = \frac{6}{10} \quad \Omega^{(3)} = \frac{3}{10}$$

“Smoothing” monitors

$$IS_1^L = \frac{13}{12}\left(\mathbf{S}_{ij}^{-1}\Delta\mathbf{Q}_{5/2} - \mathbf{S}_{ij}^{-1}\Delta\mathbf{Q}_{3/2}\right)^2 + \frac{1}{4}\left(3\mathbf{S}_{ij}^{-1}\Delta\mathbf{Q}_{5/2} - \mathbf{S}_{ij}^{-1}\Delta\mathbf{Q}_{3/2}\right)^2$$

$$IS_2^L = \frac{13}{12}\left(\mathbf{S}_{ij}^{-1}\Delta\mathbf{Q}_{7/2} - \mathbf{S}_{ij}^{-1}\Delta\mathbf{Q}_{5/2}\right)^2 + \frac{1}{4}\left(\left(\nabla\mathbf{F}\right)_{ij}^{c}\cdot\mathbf{ij} + \mathbf{S}_{ij}^{-1}\Delta\mathbf{Q}_{5/2}\right)^2$$

$$IS_3^L = \frac{13}{12}\left(\mathbf{S}_{ij}^{-1}\Delta\mathbf{Q}_{9/2} - \mathbf{S}_{ij}^{-1}\Delta\mathbf{Q}_{7/2}\right)^2 + \frac{1}{4}\left(\mathbf{S}_{ij}^{-1}\Delta\mathbf{Q}_{9/2} - 3\mathbf{S}_{ij}^{-1}\Delta\mathbf{Q}_{7/2}\right)^2$$

3. WENO-reconstruction

$$\left.\begin{aligned}\mathbf{Q}_{ij}^{L} &= \mathbf{S}_{ij}\mathcal{L}_{ij}^{L} = \mathbf{Q}_i + \sum_{k=1}^{3}\omega_k^L\, SlopeL(k) \\ \mathbf{Q}_{ij}^{R} &= \mathbf{S}_{ij}\mathcal{L}_{ij}^{R} = \mathbf{Q}_j - \sum_{k=1}^{3}\omega_k^R\, SlopeR(k)\end{aligned}\right\} \to h\left(\mathbf{Q}_{ij}^{L}, \mathbf{Q}_{ij}^{R}\right)$$

Riemann solver

Characteristic-wise quasi-FD approach = **CH-WENO-EBR scheme**

(Shu, 1994)

FS-WENO-EBR scheme

(for "L" case)

1. Split fluxes $\mathbf{F} = \mathbf{F}^{+} + \mathbf{F}^{-}$, $\mathbf{F}^{\pm} = \frac{1}{2}\left(\mathbf{F} \pm |\mathbf{A}|\mathbf{Q}\right)$ or $\left(\mathbf{F}^{\pm} = \frac{1}{2}\left(\mathbf{F} \pm \operatorname{sign}(\mathbf{A})\mathbf{F}\right)\right)$

2. Three different characteristic 2nd order reconstructions of split fluxes on the cell interface *ij*

$$\left(\mathcal{L}^{\pm}\right)_{ij}^{(k)} = \sum_{l=1}^{3}\left(\alpha_{ij,l}^{\pm}\right)^{(k)} \frac{1}{2}\left(\mathbf{S}_{ij}^{-1}\mathbf{F}_l \pm |\mathbf{\Lambda}|_{ij}\,\mathbf{S}_{ij}^{-1}\mathbf{Q}_l\right), \quad k = 1,2,3, \quad \mathbf{S}_{ij}^{-1} = \mathbf{S}_{ij}^{-1}\left(\mathbf{Q}_i, \mathbf{Q}_j\right)$$

$$\left(\alpha_{ij,1}^{\pm}\right)^{(k)} = 1 - \left(\beta_{ij,1}^{L/R}\right)^{(k)}, \quad \left(\alpha_{ij,2}^{\pm}\right)^{(k)} = \left(\beta_{ij,1}^{L/R}\right)^{(k)} - \left(\beta_{ij,2}^{L/R}\right)^{(k)}, \quad \left(\alpha_{ij,3}^{\pm}\right)^{(k)} = \left(\beta_{ij,2}^{L/R}\right)^{(k)}$$

3. Weighted sum of the reconstructions

$$\mathcal{L}_{ij}^{\pm} = \sum_{k=1}^{3}\omega_k^{\pm}\left(\mathcal{L}^{\pm}\right)_{ij}^{(k)}, \quad \omega_k^{\pm} = \omega_k^{L/R}$$ (with replacing $\mathbf{S}_{ij}^{-1}\Delta\mathbf{Q}$ with $\frac{1}{2}\left(\mathbf{S}_{ij}^{-1}\mathbf{F}_l \pm |\mathbf{\Lambda}|_{ij}\,\mathbf{S}_{ij}^{-1}\mathbf{Q}_l\right)$ in "smoothing" monitors)

4. FS-WENO-reconstruction

$$\begin{aligned}\mathbf{F}_{ij} = {} & \mathbf{S}_{ij}\mathcal{L}_{ij}^{+} + \mathbf{S}_{ij}\sum_{k=1}^{3}\omega_k^{-}\sum_{l=1}^{3}\left[\left(\alpha_{ij,l}^{-}\right)^{(k)}\frac{1}{2}\left(\mathbf{S}_{ij}^{-1}\mathbf{F}_l - |\mathbf{\Lambda}|_{ij}\,\mathbf{S}_{ij}^{-1}\mathbf{Q}_l\right)\right] \\ & + \mathbf{S}_{ij}\mathcal{L}_{ij}^{-} + \mathbf{S}_{ij}\sum_{k=1}^{3}\omega_k^{+}\sum_{l=1}^{3}\left[\left(\alpha_{ij,l}^{+}\right)^{(k)}\frac{1}{2}\left(\mathbf{S}_{ij}^{-1}\mathbf{F}_l + |\mathbf{\Lambda}|_{ij}\,\mathbf{S}_{ij}^{-1}\mathbf{Q}_l\right)\right]\end{aligned}$$

Characteristic-wise flux splitting quasi-FD approach = **FS-WENO-EBR scheme**

Remark, or the goals of verification

The reference data are the numerical results of corresponding computations on 2D or 3D Cartesian meshes.

The method for unstructured meshes results obviously inherit the properties (including possible drawbacks) of the 1D solver in use.

The verification of solvers and the choice of better ones are beyond the present study.

The main goal is
to compare "unstructured" WENO-EBR scheme with the FD WENO scheme on structured meshes

Test problems

1D Test problems

— Burgers equation

— 1D Riemann problem
(Zhmakin A.I., Fursenko A.A. , *U.S.S.R. Comput. Maths. Math. Phys*., 20(4), 1980)

— Blast Wave problem
(Woodward P., Colella P., *J. Comput. Phys*., 54(1), 1984)

2D Test problems

— 2D Riemann problem
(Liska R., Wendroff B. *SIAM J. Sci. Comput.*, 25(3), 2003)

— 2D Mach 3 Wind Tunnel with a Step
(Woodward P., Colella P., *J. Comput. Phys*., 54(1), 1984)

Meshes in use

— Uniform Cartesian mesh
(3D uniform mesh identical to 1D uniform mesh)

— Unstructured mesh
(2D – triangle mesh, 3D – tetrahedral mesh)

1D problems are calculated on 3D uniform Cartesian mesh and 3D unstructured mesh

Time integration

— Explicit 3-step scheme of Runge-Kutta
(Shu C.-W., Osher S., J. Comput. Phys., 77(2), 1988)

— CFL = 0.4

Reconstructed variables

— Reconstruction of the characteristic variables (CH-WENO-EBR)

— Reconstruction of the split fluxes (FS-WENO-EBR)

Determination of the characteristic variables

— the Jacobian is based on the velocity normal to the cell face

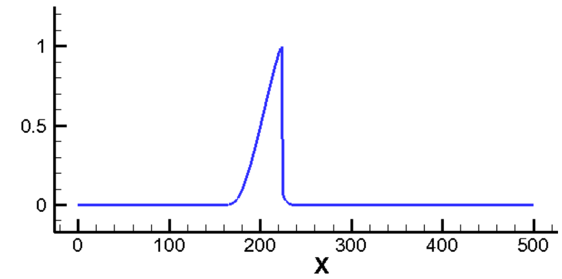
— the Jacobian is based on the velocity projected on the corresponding edge

Burgers equation

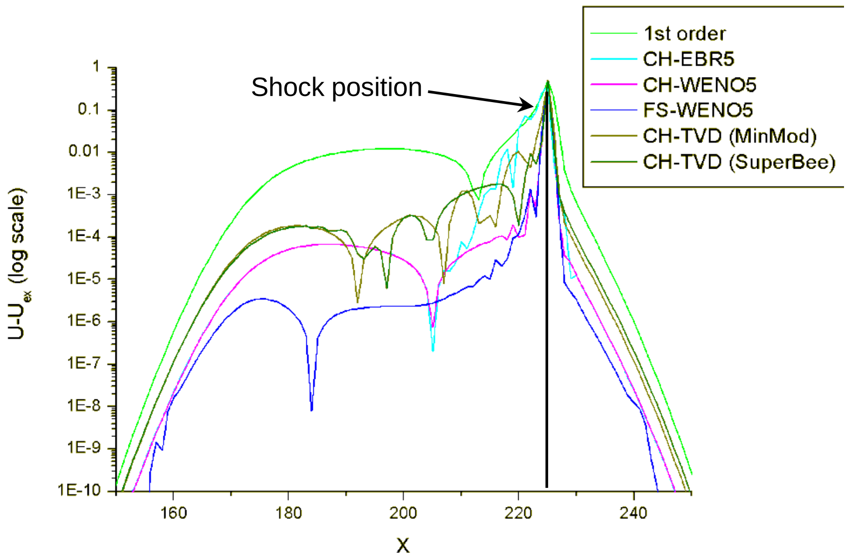
$T_{\max} = 25$

Problem formulation

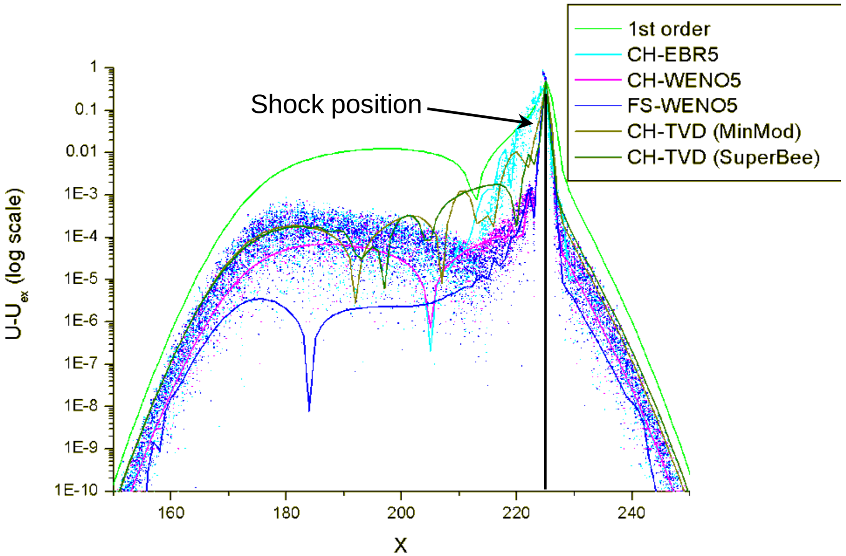
$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{u^2}{2}\right) = 0 \\ u|_{t=0} = \exp\left(-\ln(2)\dfrac{(x-200)^2}{12^2}\right) \end{cases}$$



Uniform mesh



3D unstructured mesh



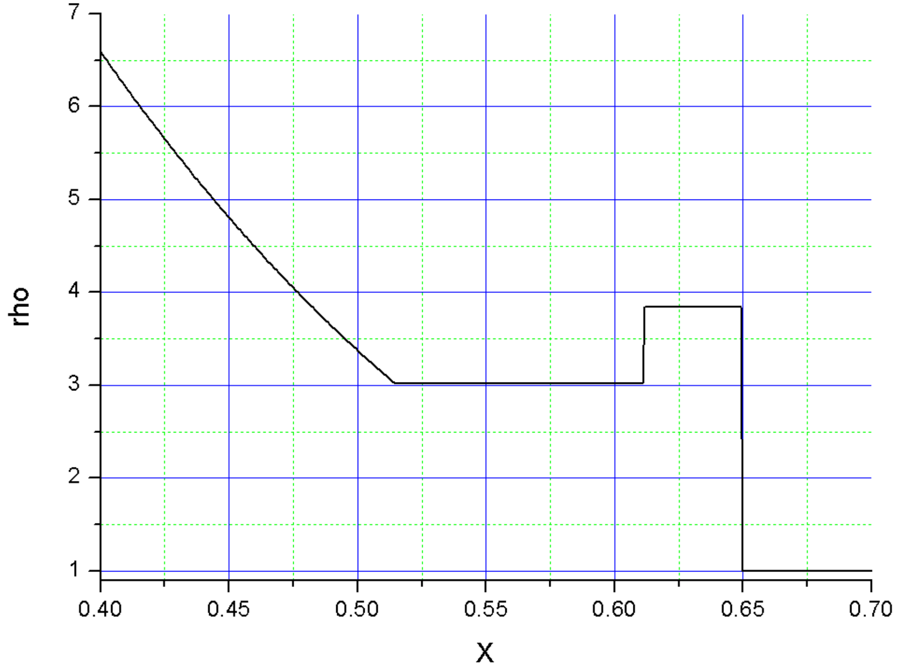
1D Riemann problem

Problem formulation

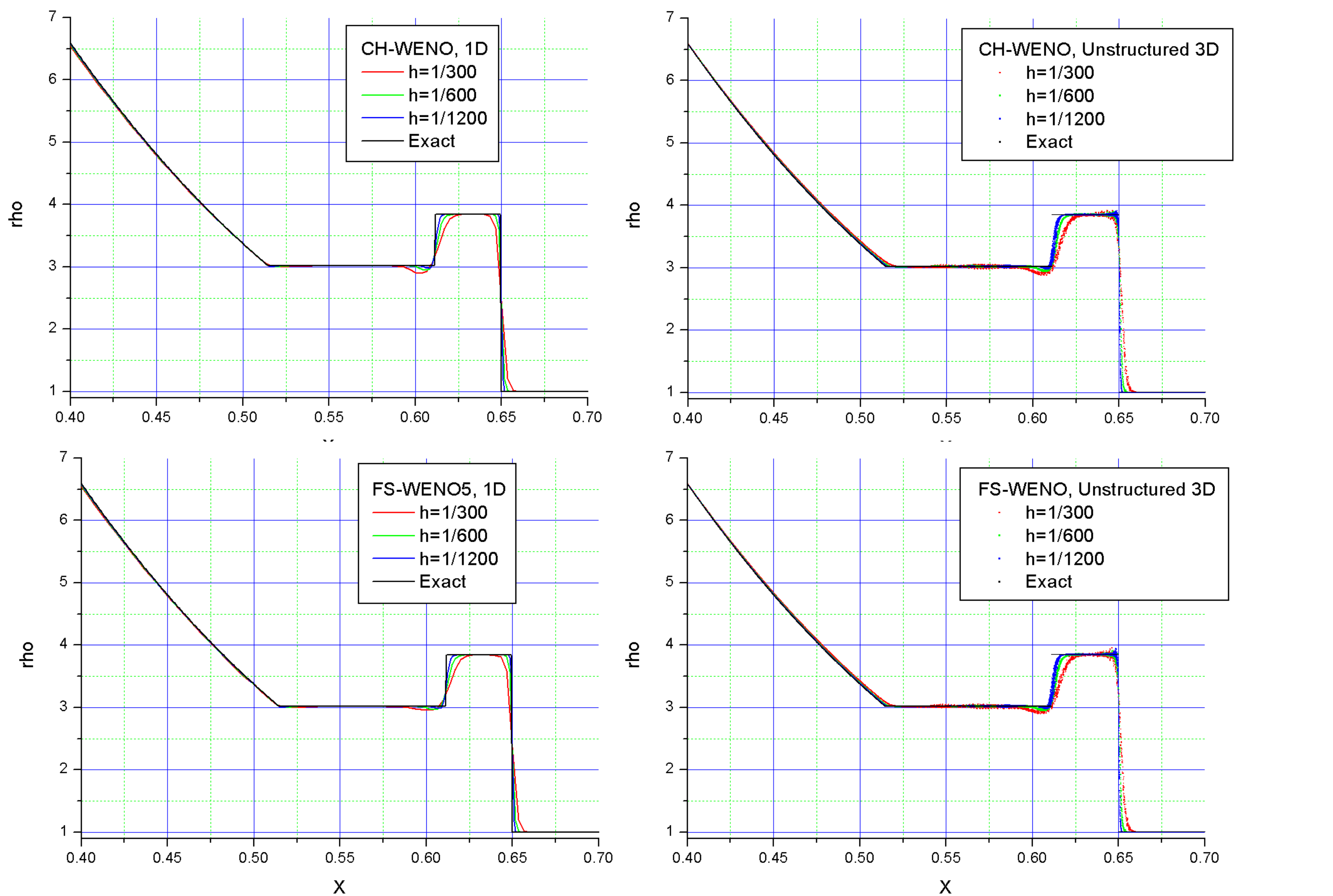
- Euler equations (γ = 5/3)
- Initial condition

$$\begin{cases} \rho_L = 8, & \rho_R = 1 \\ u_L = 0 & u_R = 0 \\ p_L = 480 & p_R = 1 \end{cases}$$

Exact solution $T_{\max} = 4$



Verification of WENO-EBR Scheme



Blast wave problem

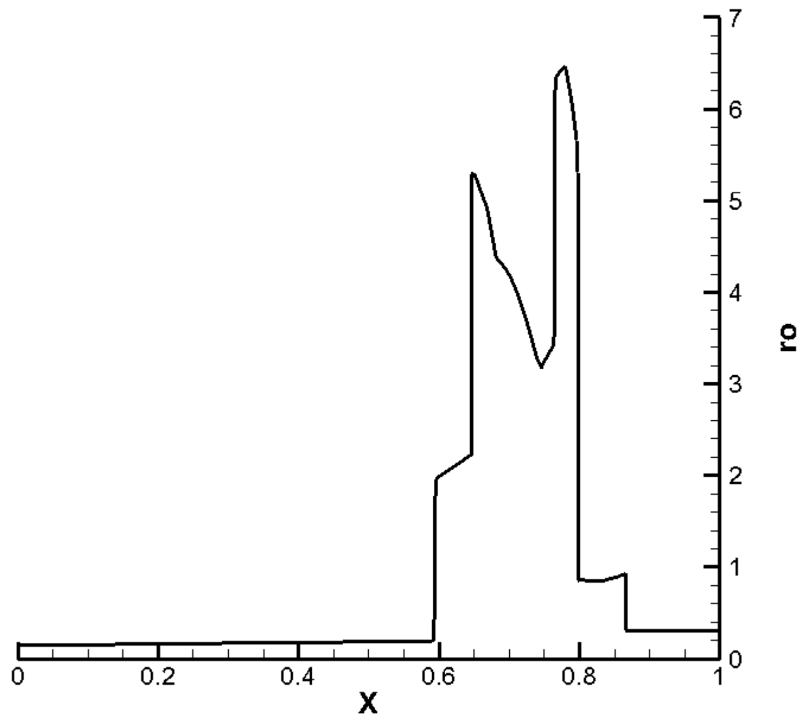
Problem formulation

- Euler equations (γ = 7/5)
- Initial condition

$$\rho = 1, \quad u = 0, \quad p = \begin{cases} 1000, & x < 0.1 \\ 0.01, & 0.1 < x < 0.9 \\ 100, & x > 0.9 \end{cases}$$

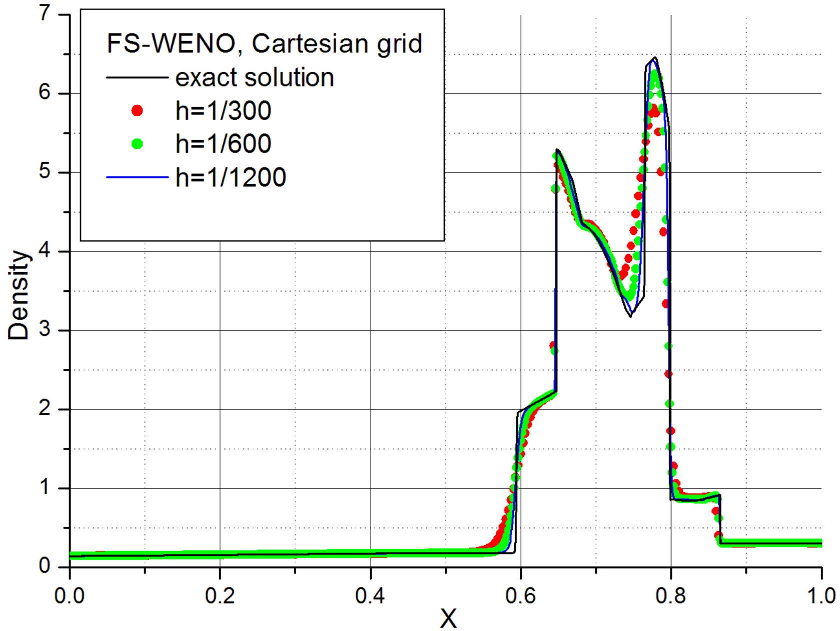
- Reflection BC for x=0 and x=1

Reference solution $T_{\max} = 0.038$

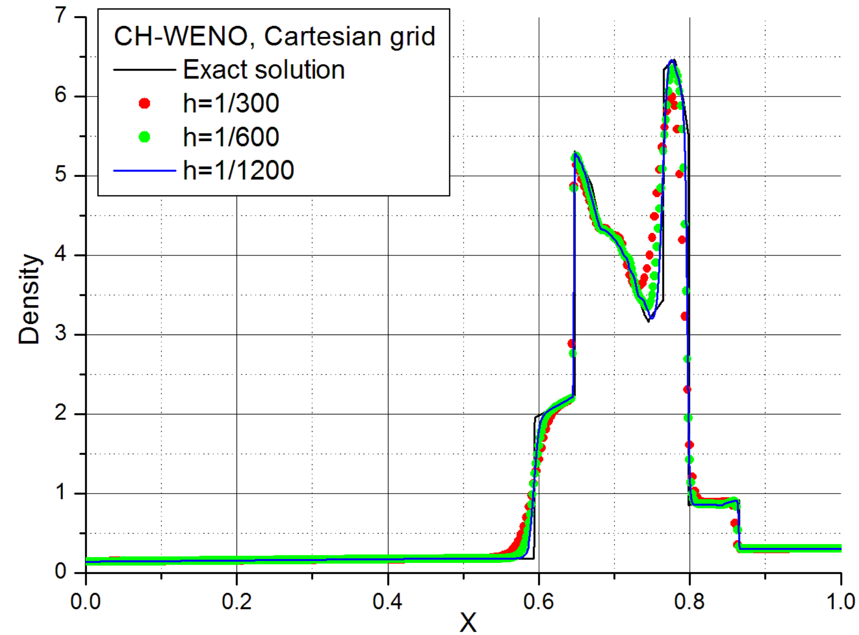


Blast wave problem

Cartesian uniform mesh



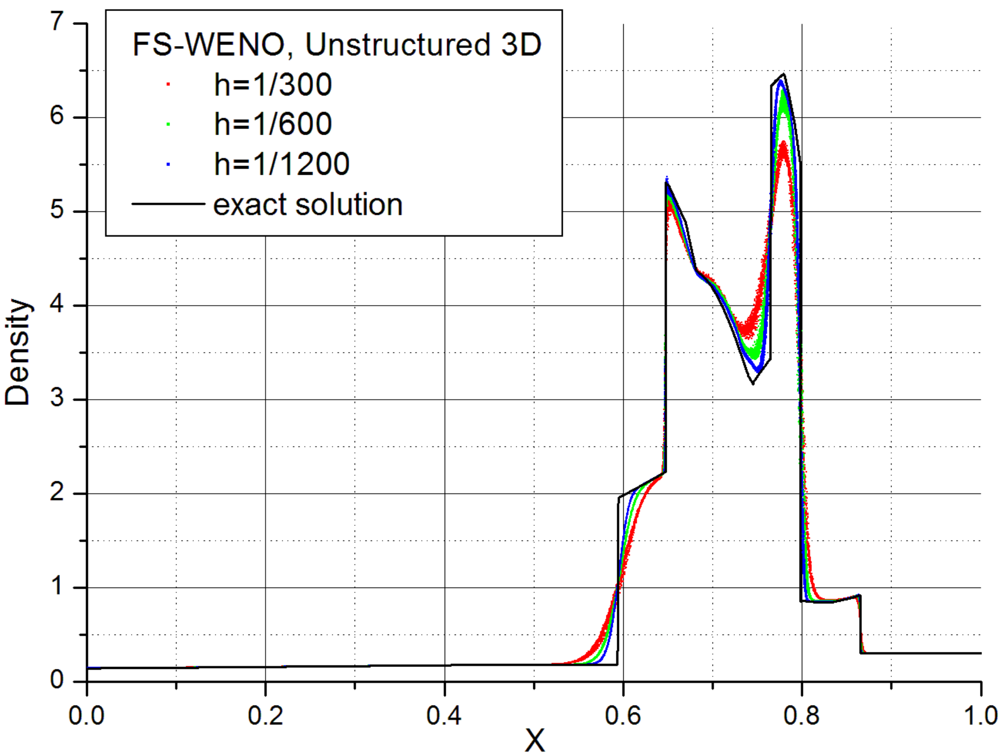
FS-WENO scheme



CH-WENO scheme

Blast wave problem

3D unstructured mesh



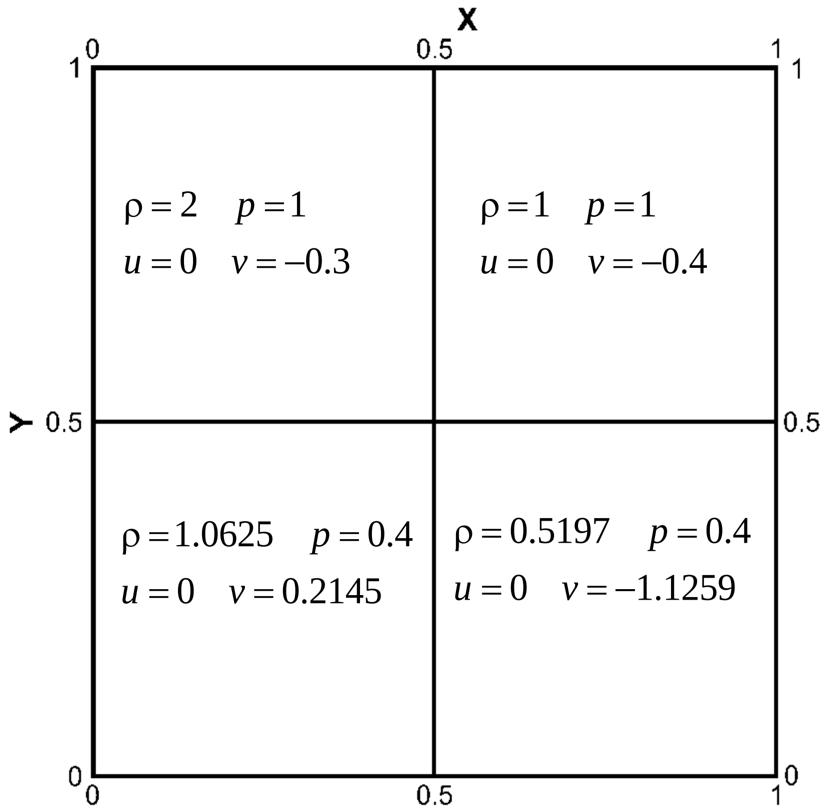
FS-WENO scheme

CH-WENO scheme

2D Riemann problem

Problem formulation

- Euler equations (γ = 7/5)
- Initial condition



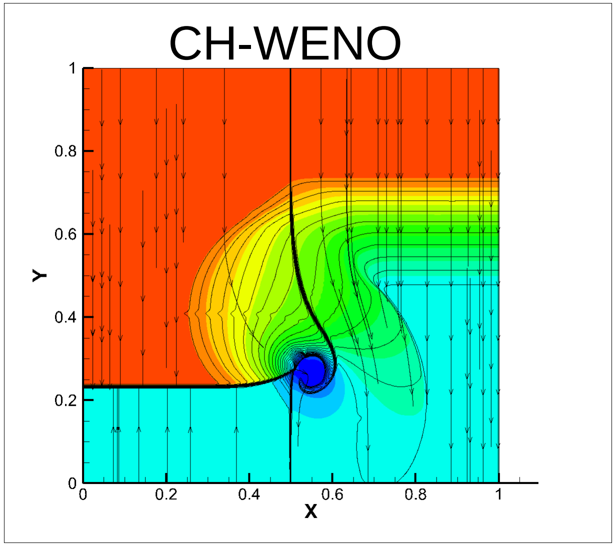
$T_{max} = 0.3$

- Reflection BC for x=0 and x=1
- Input/Output BC for y=0/y=1

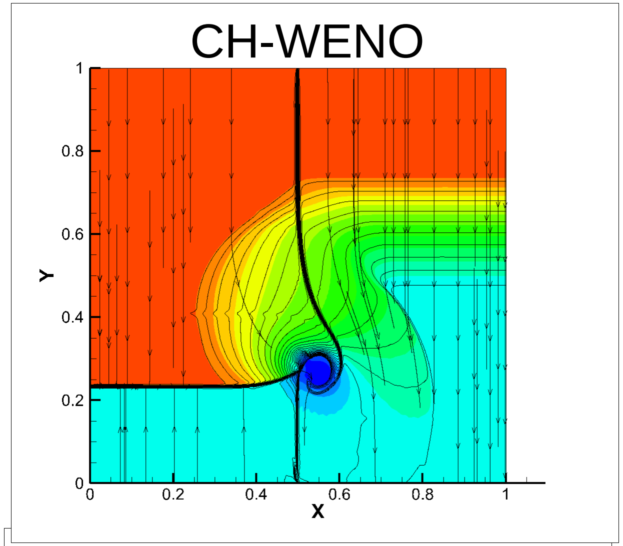
2D Riemann problem

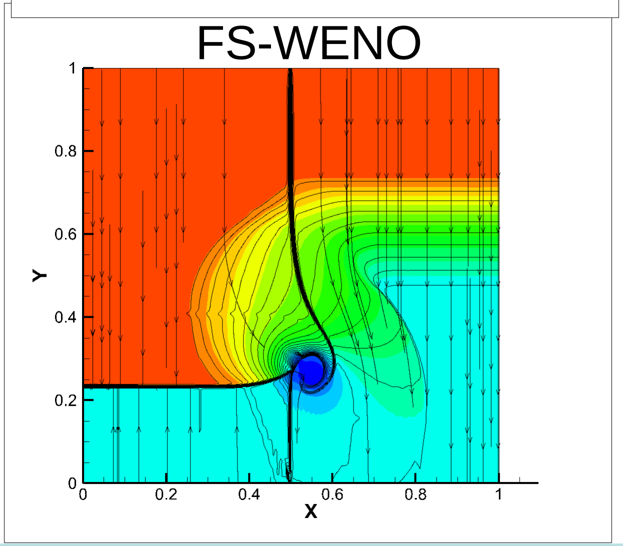
Cartesian mesh

Square cells



Hexagonal cells

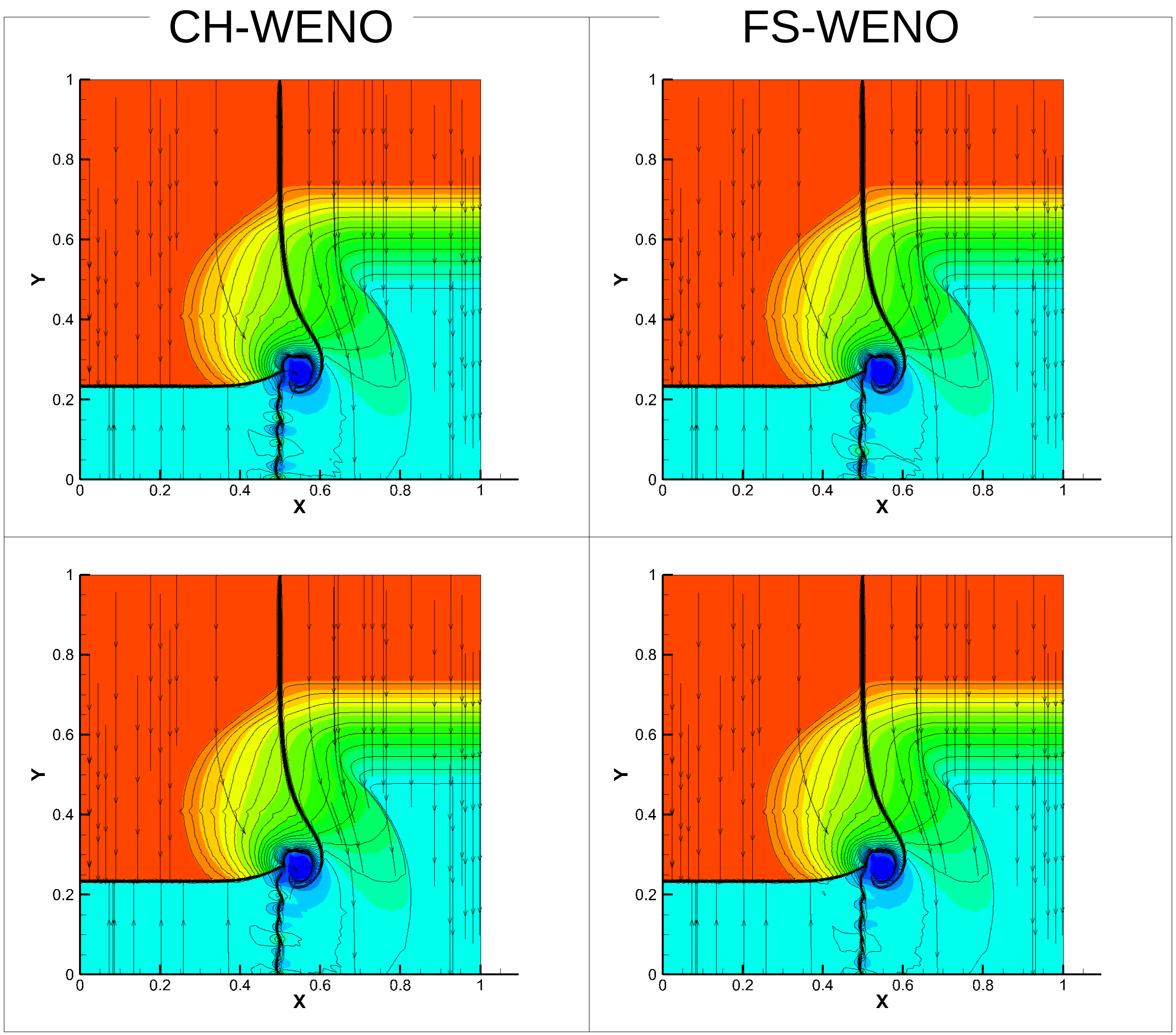




Mesh size h=1/400

2D Riemann problem

2D unstructured mesh



Normal velocity in Jacobian

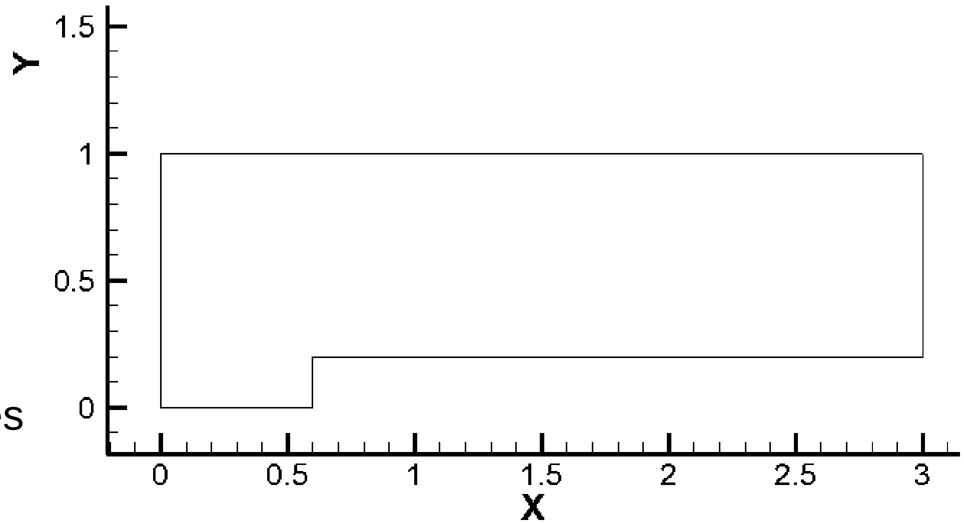
Edge-oriented velocity in Jacobian

2D Mach 3 Wind Tunnel with a Step

Problem formulation

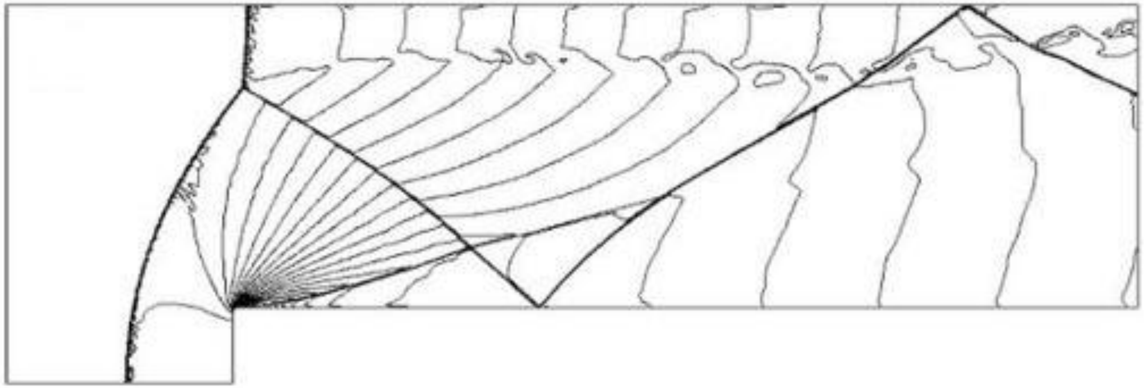
- Euler equations (γ = 7/5)
- Initial condition

$$\rho = 1.4 \quad p = 1 \quad u = 3 \quad v = 0$$
$$\gamma = 1.4 \quad T_{max} = 4$$



- Reflection BC on horizontal boundaries
- Input/Output BC for x=0/x=3

Reference solution



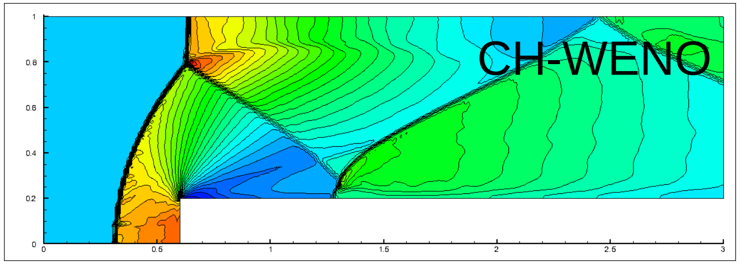
Woodward P., Colella P.
The numerical simulation of two-dimensional fluid flow with strong shocks // *Journal of Computational Physics*. 1984. V.54, PP.115-173.

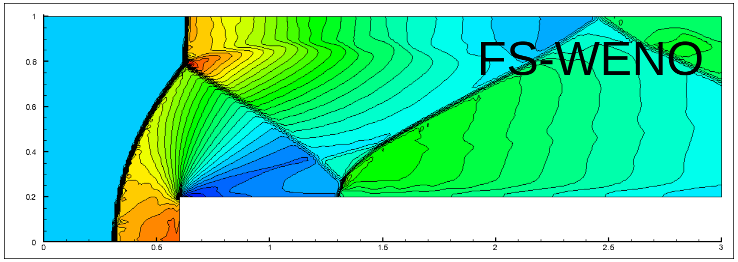
Verification of WENO-EBR Scheme

2D Mach 3 Wind Tunnel with a Step

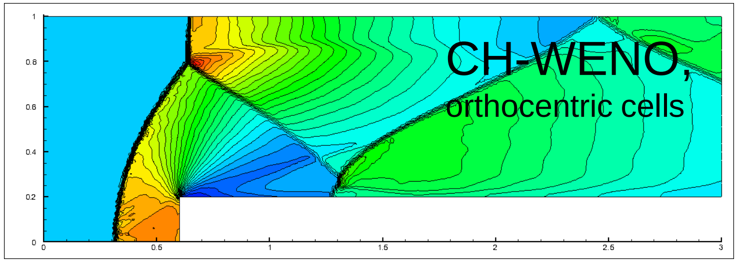
2D results. Density

Cartesian mesh

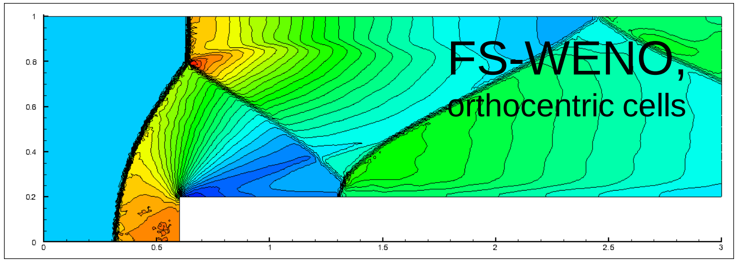




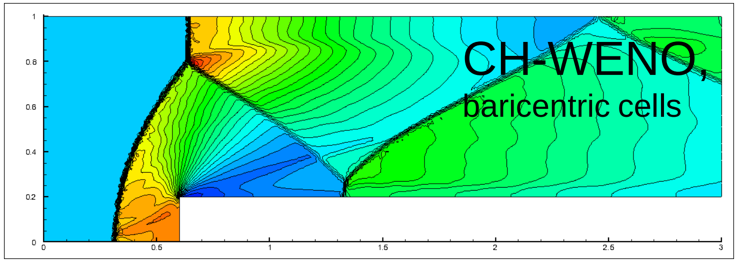
Unstructured triangle mesh



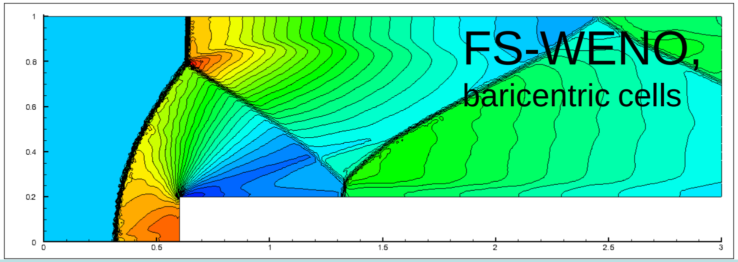
Mesh size h=1/80



Characteristics in the normal direction



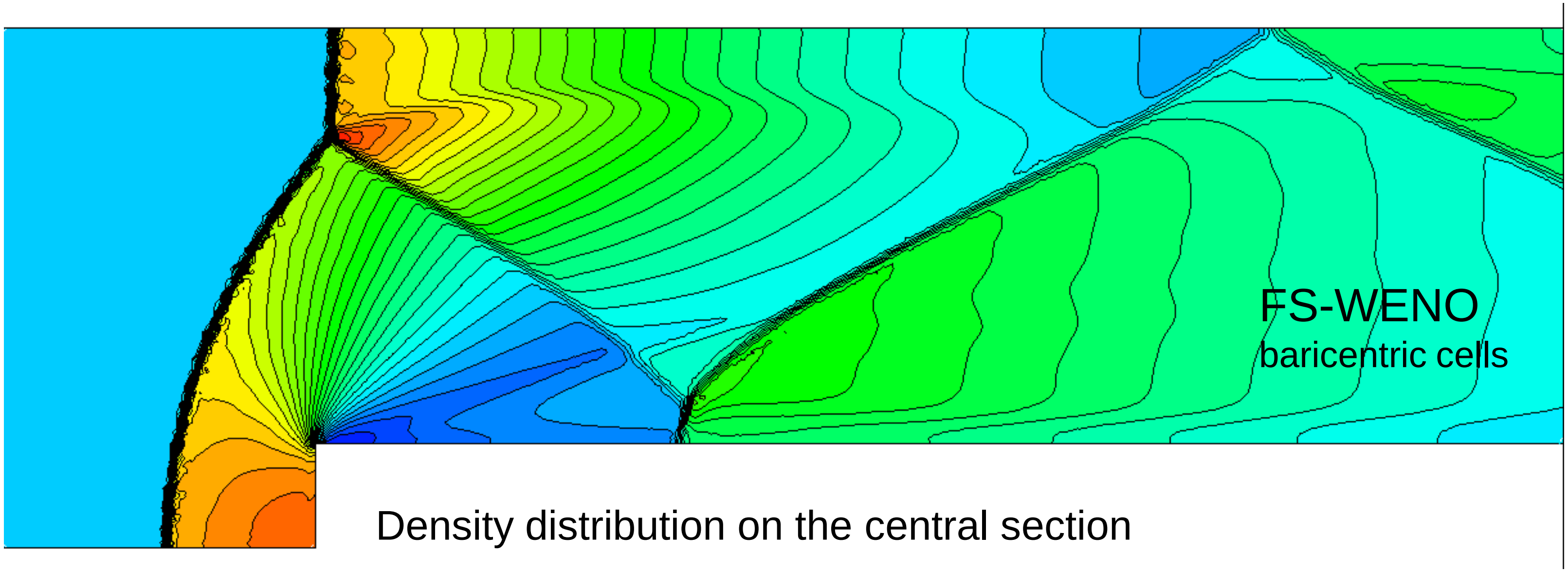
The 1st-order accuracy on the boundary



2D Mach 3 Wind Tunnel with a Step

3D results

Unstructured tetrahedral mesh, 20 points in transverse direction



Density distribution on the central section

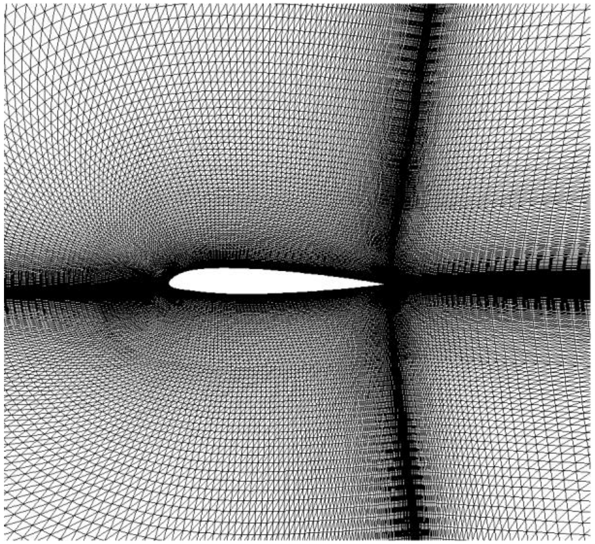
Verification of WENO-EBR Scheme

Flow around NACA23101 Airfoil

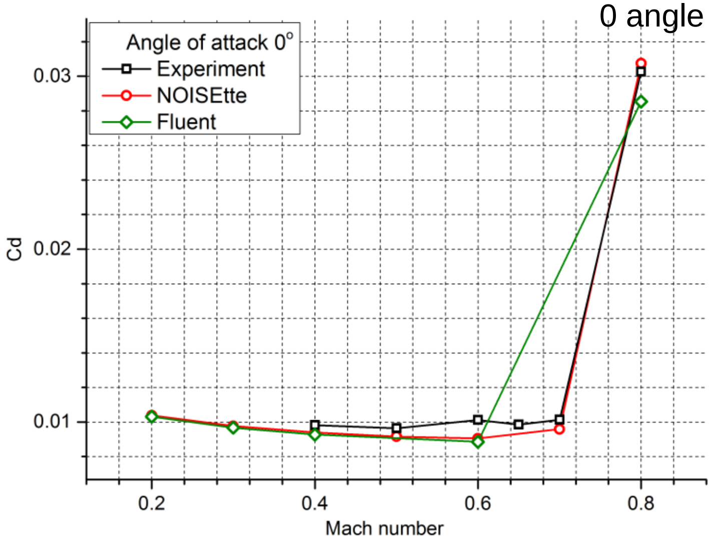
RANS equations + SA turbulent model

Reynolds number: $2 \cdot 10^6$

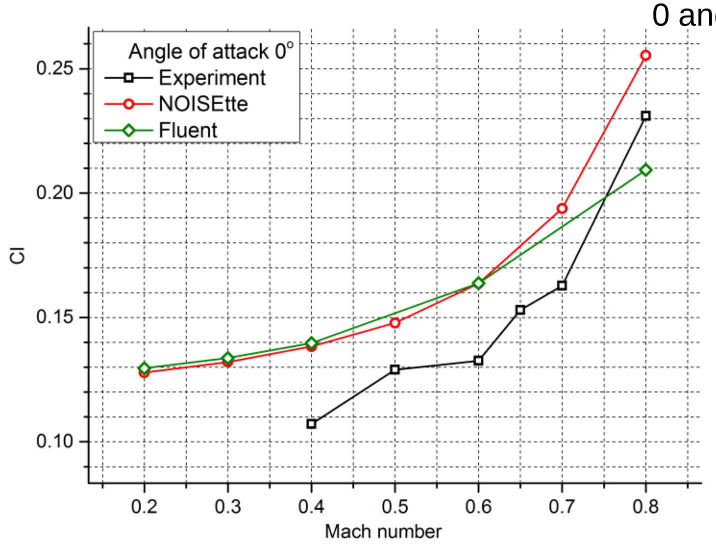
Triangle structured mesh
100 600 nodes, 200 000 elements



Drag coefficient vs Mach number
0 angle of attack



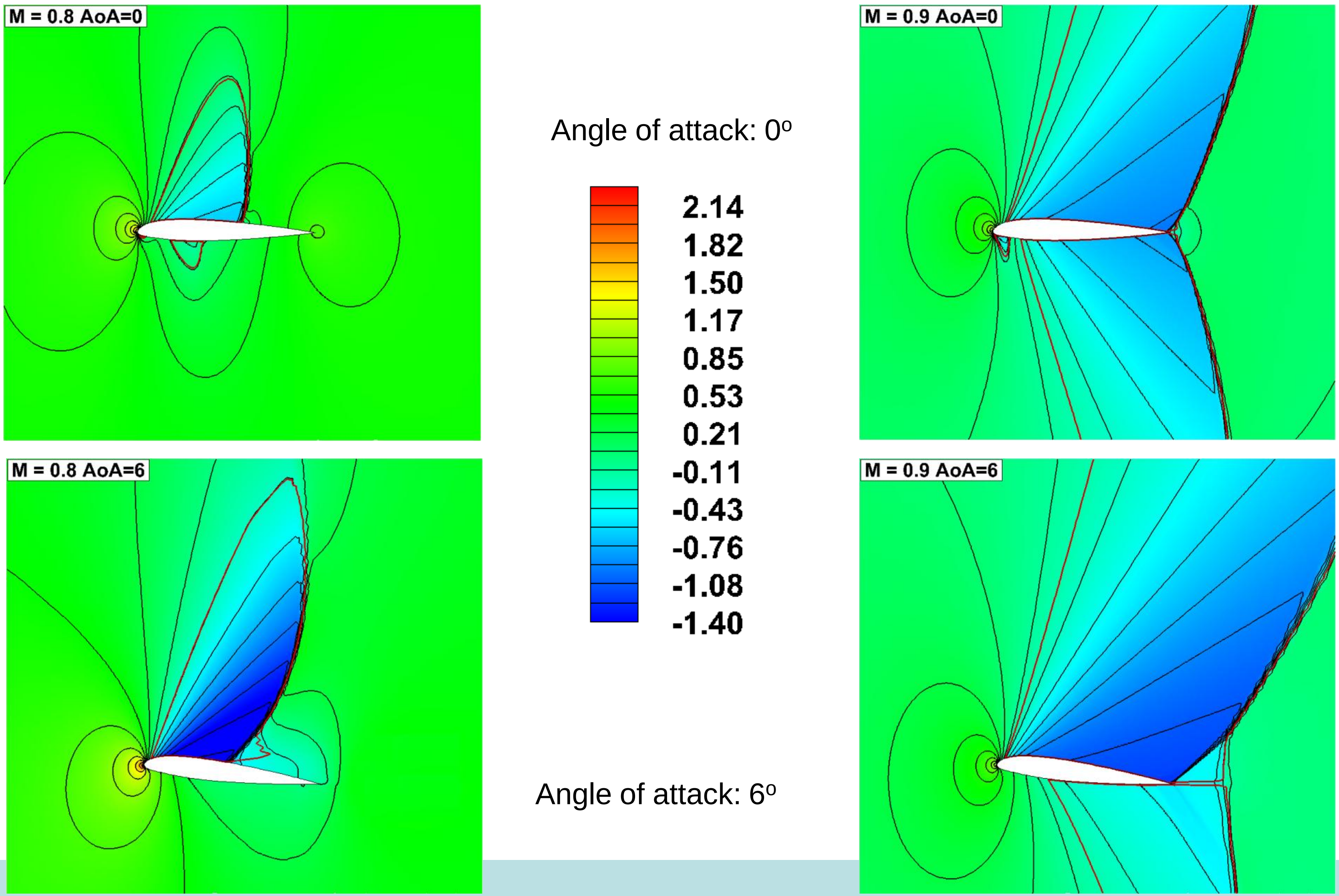
Lift coefficient vs Mach number.
0 angle of attack



Verification of WENO-EBR Scheme

Flow around Airfoil NACA23101

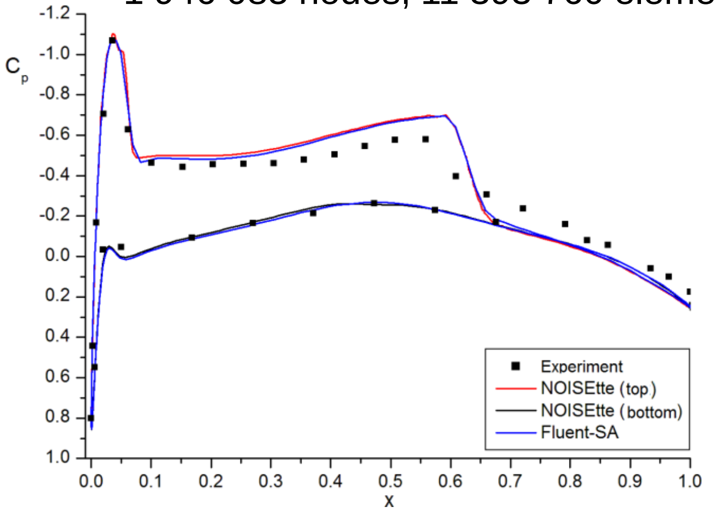
Mach number: 0.8 | Pressure coefficient distribution and sonic line | Mach number: 0.9



Verification of WENO-EBR Scheme

Flow around ONERA M6 3D Wing

Tetrahedral unstructured mesh
1 946 085 nodes, 11 393 760 elements

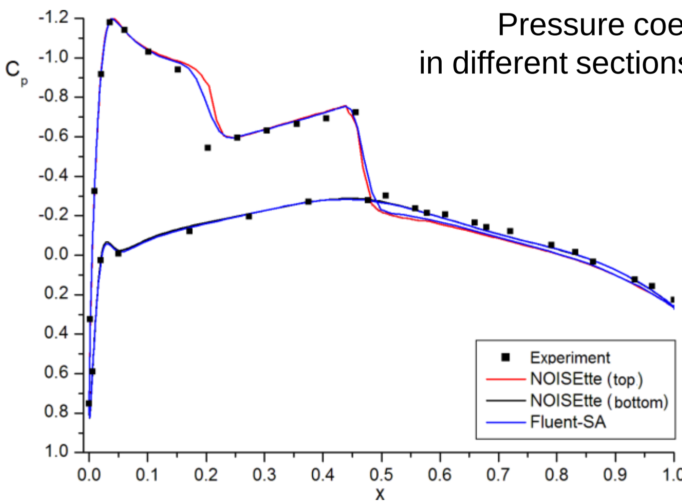


RANS equations + SA turbulence model

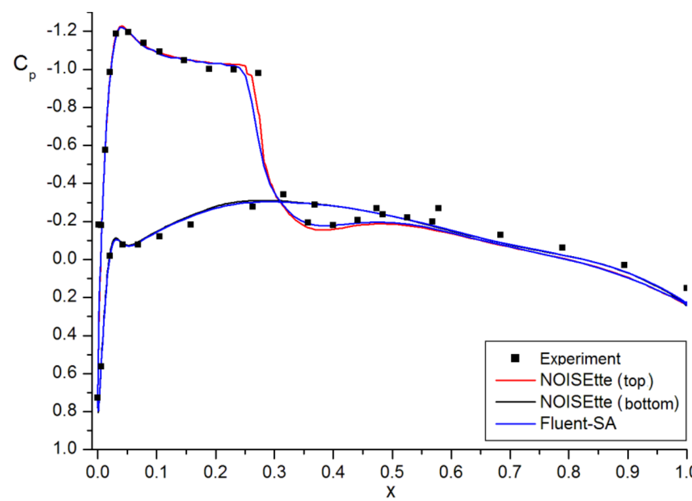
Reynolds number: $1.9 \cdot 10^7$, Mach number: 0.84

	CL	CD
Fluent-SA	0.2705	**0.0175**
Fluent-SST	0.2664	**0.0170**
NOISEtte (WENO-EBR)	**0.2675**	**0.0168**

Lift and drag coefficients



Pressure coefficient
in different sections of the wing



Summary & Conclusions

We have developed **WENO-EBR scheme**.
It can be considered as a **light WENO scheme** for unstructured meshes.

It has been verified on a set of benchmark problems
and confirmed its accuracy close to the classical FD WENO on structured meshes.

WENO-EBR scheme has been implemented in **in-house code NOISEtte**.

Now we use it successfully for simulating aerodynamics and aeroacoustics of:

- transonic flows over wedge-shaped body with a backward step (a model investigated experimentally in wind tunnel);
- transonic and supersonic flows around 2D and 3D airfoils;
- transonic flows around single rotating blade.

Thank you for your attention!