Third International Workshop
"Computational Experiment in Aeroacoustics"

Real accuracy of monotone and non-monotone shock capturing schemes for flows with shock waves

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Svetlogorsk, Russia
September 24-27, 2014.

Finite difference schemes for hyperbolic systems of conservation laws

$$\frac{\partial \boldsymbol{u}(x,t)}{\partial t} + \frac{\boldsymbol{f}(\boldsymbol{u}(x,t))}{\partial x} = 0, \tag{1}$$

Finite difference scheme approximating the system (1) is

$$\frac{\boldsymbol{v}_j^{n+1} - \boldsymbol{v}_j^n}{\tau} + \frac{\overline{\boldsymbol{f}}_{j+1/2}^{\,n} - \overline{\boldsymbol{f}}_{j-1/2}^{\,n}}{h} = 0, \tag{2}$$

where $\overline{\boldsymbol{f}}_{j+1/2}^{\,n} = \overline{\boldsymbol{f}}\big(\boldsymbol{v}_{j-l+1}^n, \boldsymbol{v}_{j-l+2}^n, \ldots, \boldsymbol{v}_{j+l}^n, \lambda\big)$ is a vector-function of numerical flux consistent with the flux $\boldsymbol{f}$ of the system (1) in the following sense:

$$\boldsymbol{f}(\boldsymbol{u}, \boldsymbol{u}, \ldots, \boldsymbol{u}, \lambda) \equiv \boldsymbol{f}(\boldsymbol{u})$$

The Courant stability condition is

$$\lambda = \frac{\tau}{h} < \frac{1}{\max\limits_{k,j,n} |a_k(\boldsymbol{v}_j^n)|},$$

where $a_k(\boldsymbol{u})$ are eigenvalues of the Jacobi matrix $\boldsymbol{f}_{\boldsymbol{u}}$ of the system (1).

Godunov's barrier

There are no monotone difference scheme
(with smooth numerical flux function)
higher than of the first order

Godunov S.K. A Difference Scheme for Numerical Solution of Discontinuous Solutions of Hydrodynamic Equations // Mat. Sb, 1959, Vol. 47, №3, pp. 271–306.

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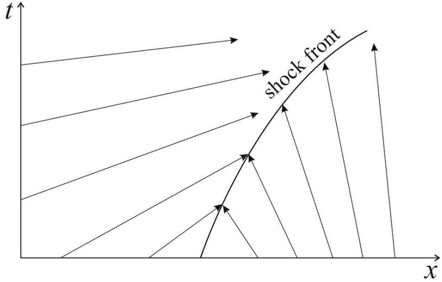
Attempts to overcome Godunov's barrier

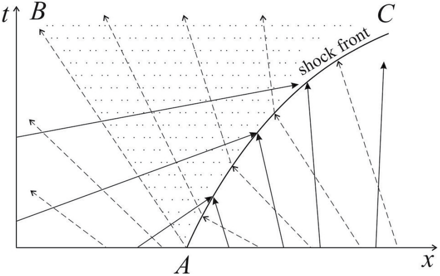
TVD-like schemes

- FCT, *Boris, Book*, 1975.
- *Kolgan*, 1978.
- MUSCL, *Van Leer*, 1979.
- TVD schemes, *Harten*, 1983.
- ENO schemes, *Harten, Osher*, 1987.
- NED schemes, *Tadmor*, 1990.
- WENO schemes, *Liu, Osher, Chan*, 1994; *Jiang, Shu*, 1996.

Domain of influence of shock waves



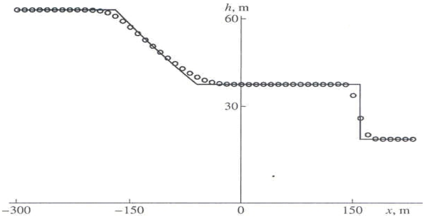
Characteristic field for scalar conservation law



Characteristic fields for a system of two conservation laws

Why erroneous view that TVD-like schemes should maintain an increased order of convergence in all smooth parts of generalized solutions did exist for so long time?

1. In the most part of papers on numerical schemes of higher accuracy, the schemes were tested using a Riemann problem solution, where only stationary shock waves emerge with a constant flow behind their fronts.



a Riemann problem solution

2. For a scalar conservation law there is no characteristic field crossing a shock wave. As a result, the domain of influence of the shock wave coincides with its front.

Real accuracy of TVD schemes

In [1–3] it was shown that TVD schemes of formally high accuracy decrease their order of convergence down to the first order in domains of influence of non-stationary shock waves.

1. *Ostapenko V.V.* On convergence of difference schemes behind a front of a non-stationary shock wave // Zh. vych. matem. i matem. fiz. (1997) Vol. 37, No. 10, pp. 1201–1212.
2. *Casper J., Carpenter M.N.* Computational consideration for the simulation of shock-induced sound // SIAM J. Sci. Comput. (1998) Vol. 19, No. 1, pp. 813-828.
3. *Engquist B., Sjogreen B.* The convergence rate of finite difference schemes in the presence of shocks // SIAM J. Numer. Anal. (1998), Vol. 35. pp. 2464-2485.

Finite difference schemes

- **MacCormack R.W.** The effect of viscosity in hypervelocity impact cratering // AIAA Paper 69–354 (1969).
- **Rusanov V.V.** Difference schemes of third order for shock capturing of discontinuous solutions // Soviet Doklady (1968), Vol. 180, № 6, pp. 1303–1305.
- **Harten A.** High resolution schemes for hyperbolic conservation laws // J. Comp. Phys. (1983), Vol. 49, pp. 357–393.
- **Jiang G.-S., Shu C.-W.** Efficient implementation of weighted ENO schemes // J. Comput. Phys. (1996), Vol. 126, pp. 202–228.

$$TV(u) = \sum_i |u_{i+1} - u_i|$$

For TVD schemes

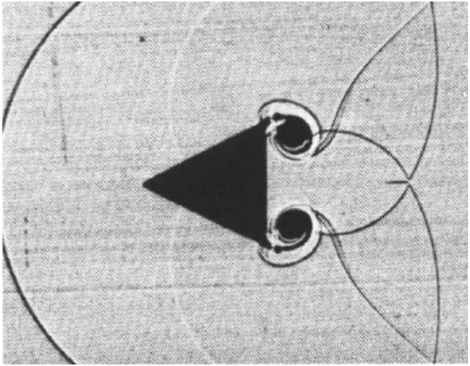
$$TV(u^{n+1}) \le TV(u^n)$$

For ENO/WENO schemes of r-th order

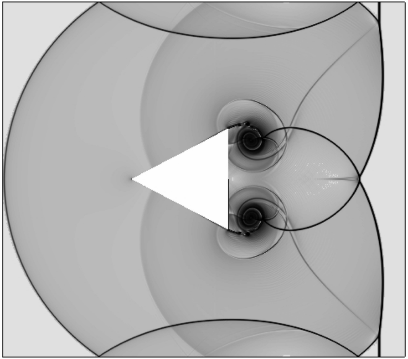
$$TV(u^{n+1}) \le TV(u^n) + O(h^r)$$

An example of numerical simulation of unsteady supersonic flow with 5th order WENO scheme

Shock wave diffraction on triangular body



Experimental shadowgraph



Numerical schlieren vizualization

Periodic initial value problem for shallow water equations

$$\begin{cases} H_t + q_x = 0 \\ q_t + (qu + gH^2/2)_x = 0 \end{cases} \tag{3}$$

$H(t,x)$ is water depth, $q(t,x)$ is flow rate, $u(t,x) = q/H$ is velocity.

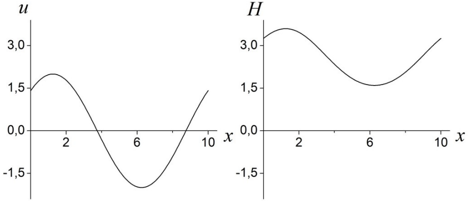
Let us consider the initial value problem for (3):

$$u(x,0) = \beta \sin\left(\frac{2\pi x}{X} + \frac{\pi}{4}\right)$$

$$H(x,0) = \frac{(u(x,0)+\theta)^2}{4g} =$$

$$= \frac{1}{4g}\left(\beta \sin\left(\frac{2\pi x}{X} + \frac{\pi}{4}\right) + \theta\right)^2$$

$$X = \theta = 10,\ \beta = 2.$$



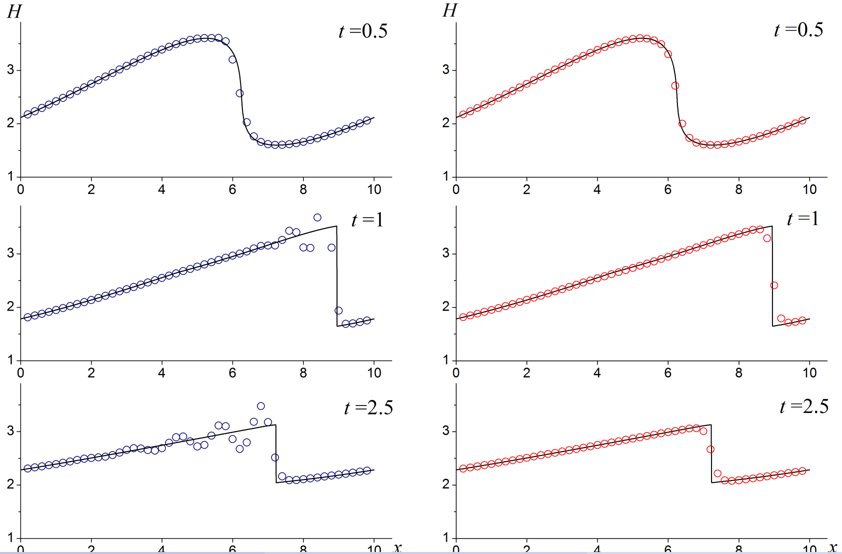
Initial values of Riemann invariants are

$$w_1(x,0) = -\theta = \text{const}, \quad w_2(x,0) = 2u(x,0) + \theta,$$

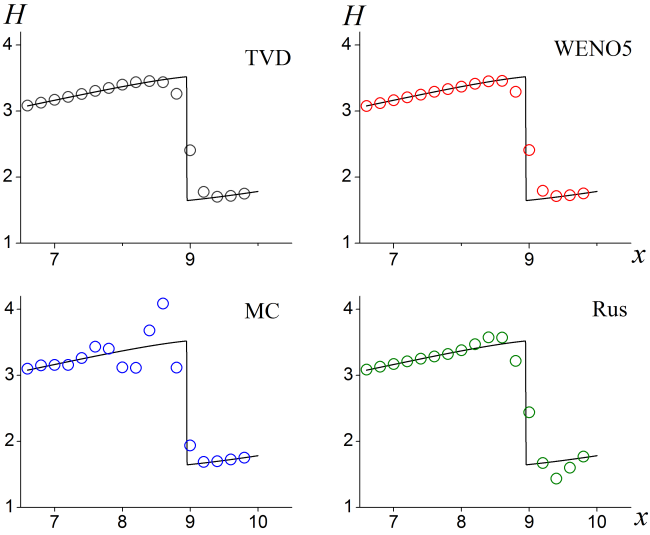
$$w_1 = u - 2c, \quad w_2 = u + 2c, \quad c = \sqrt{gH}.$$

Comparison of “exact” and numerical solutions

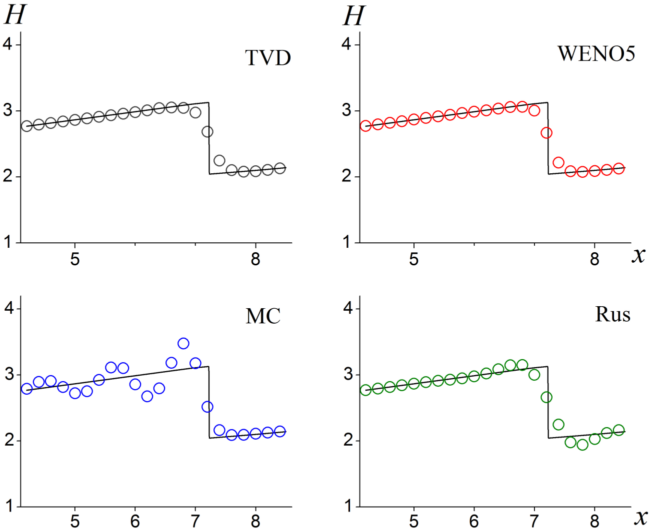
MacCormack scheme and 5th order WENO scheme on the interval $[iX, (i+1)X]$ at $i = 0$, $h = 0.2$



Finite difference solutions near shock wave front, $t = 1,\ h = 0.2$



Finite difference solutions near shock wave front, $t = 2.5,\ h = 0.2$



The Runge method for evaluation of the local accuracy of finite difference schemes

Let us consider a consequence of computational grids

$$\Omega_i = \{(x_j^i, t_n^i):\ x_j^i = jh_i,\ t_n^i = n\tau_i\}$$

$$h_0 = h,\ \tau_0 = \tau, \quad h_i = h/2^i,\ \tau_i = \lambda h_i = \tau/2^i$$

The finite difference solution $\boldsymbol{v}_j^n$ *converges with* r*-th order in the point* $(x,t) \in \Omega$ *to the solution* $\boldsymbol{u}$ *of differential equation if for the consequence of grid functions* $(\boldsymbol{v}_{h_i})_j^n$ *is valid that* $\boldsymbol{v}_{h_i}(x,t) - \boldsymbol{u}(x,t) = \boldsymbol{c}h_i^r + o(h_i^r)$ *where the vector-function* $\boldsymbol{c}$ *does not depend on* h_i*.*

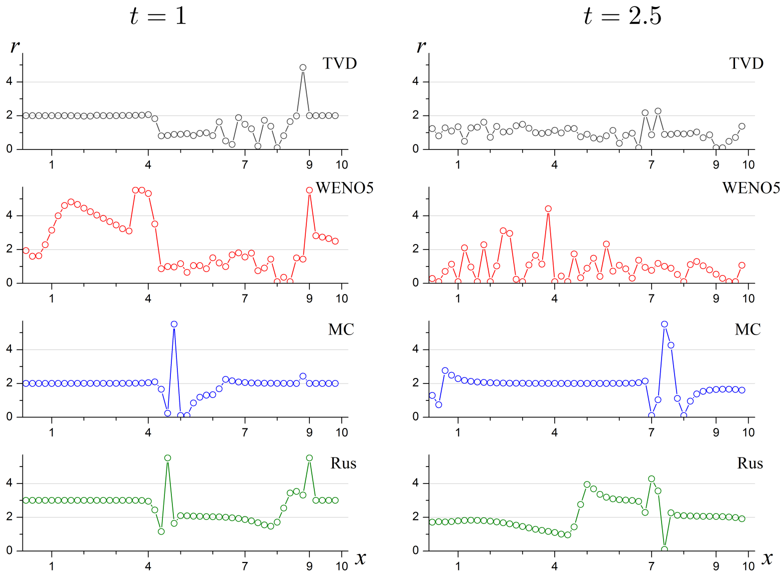
Let be $h_1 = h,\ \ h_2 = h/2,\ \ h_3 = h/4$
Then

$$r = \log_2 \frac{|\boldsymbol{v}_{h_1} - \boldsymbol{v}_{h_2}|}{|\boldsymbol{v}_{h_2} - \boldsymbol{v}_{h_3}|}$$

Local orders of convergence in different points can be significantly different and do not coincide with the formal order of scheme on smooth solutions.

Local orders of convergence, $h = 0.004$

each 50th point is shown



Definition of integral convergence

Let us specify a number $a \in \mathbb{R}$ and consider the integrals

$$\boldsymbol{U}^a(t,x) = \int\limits_x^b \boldsymbol{u}(t,y)dy, \quad \boldsymbol{V}^a_{h_i}(t,x) = \int\limits_x^a \boldsymbol{v}_{h_i}(t,y)dy$$

Definition. *The consequence of finite difference solutions* $\boldsymbol{v}_{h_i}(t,x)$ *converges on the interval* $[x,a] \subset \mathbb{R}$ *with the order* R *where* $0 < R \leq 2$ *to the solution of differential equations* $\boldsymbol{u}(t,x)$ *if*

$$\boldsymbol{V}^a_{h_i}(t,x) - \boldsymbol{U}^a(t,x) = \boldsymbol{C}h_i^R + o\left(h_i^R\right),$$

where the vector-function $\boldsymbol{C}$ *does not depend on* h_i*.*

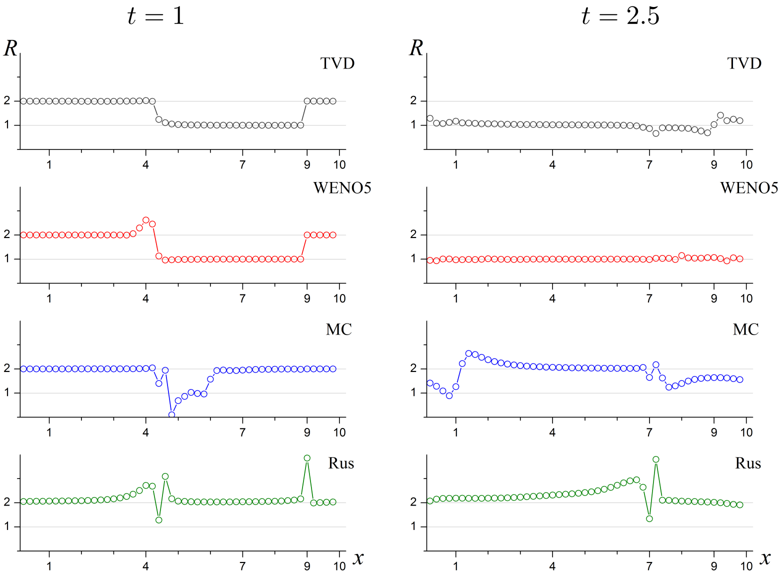
$$h_1 = h, \quad h_2 = h/2, \quad h_3 = h/4$$

$$\delta \boldsymbol{V}_i = \boldsymbol{V}^a_{h_i} - \boldsymbol{V}^a_{h_{i+1}} = \boldsymbol{C}(h_i^R - h_{i+1}^R) + o(h^R), \quad i = 1,2.$$

$$\frac{|\delta \boldsymbol{V}_1|}{|\delta \boldsymbol{V}_2|} = \frac{h_1^R - h_2^R}{h_2^R - h_3^R} = 2^R \quad \Rightarrow \quad R = \log_2 \frac{|\delta \boldsymbol{V}_1|}{|\delta \boldsymbol{V}_2|}$$

Integral orders of convergence, $h = 0.004$

each 50th point is shown

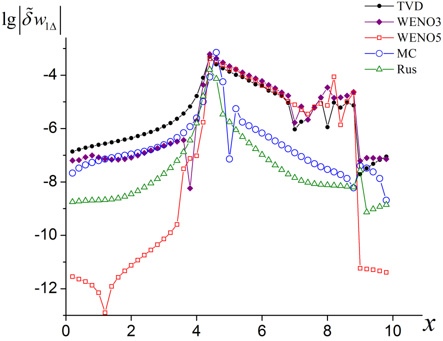


Relative errors of evaluation of Riemann invariants at $t = 1$

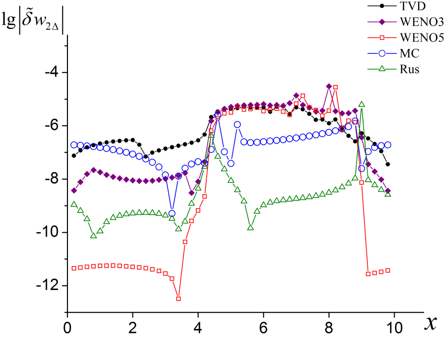
$$\bar{\delta}\varphi_{h_i} = \varphi_{h_i} - \varphi_{h_{i+1}}, \quad \delta\varphi_h = \frac{\bar{\delta}\varphi_{h_1}}{1 - |\bar{\delta}\varphi_{h_2}|/|\bar{\delta}\varphi_{h_1}|} \Rightarrow \Delta\varphi_h = \frac{\delta\varphi_h}{\varphi_h}$$

$h = 0.004$

$w_1 = u - 2\sqrt{gh}$

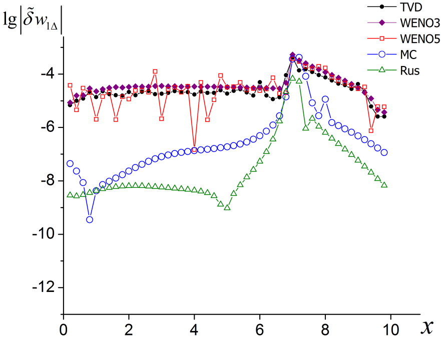


$w_2 = u + 2\sqrt{gh}$

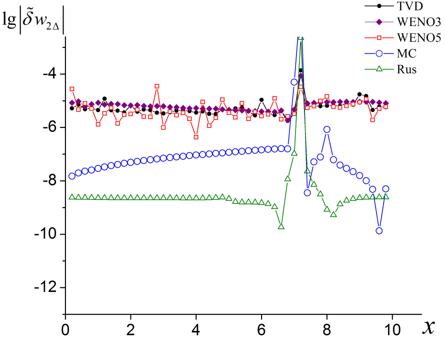


Relative errors of evaluation of Riemann invariant at $t = 2.5$

$w_1 = u - 2\sqrt{gh}$



$w_2 = u + 2\sqrt{gh}$



An alternative in the theory of finite difference schemes

Is it not possible

to localize a shock wave front with higher accuracy
and, at the same time,
maintain an increased order of convergence
in the domain of influence of the shock wave?

THANK YOU FOR YOUR ATTENTION!