Simulation of the Taylor-Green Vortex Decay Using the DRP Schemes

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Contents

Goals and methods

Objective: Simulation of aeroacoustic flows in the presence of viscous structures, including sound generation by turbulent eddies.
Available tool: Dispersion-Relation-Preserving (DRP) schemes based on high-order approximation coupled with spectral optimization.

- *C.K.W. Tam, J.C. Webb.* Dispersion-relation-preserving finite difference schemes for computational acoustics. *J. Comp. Phys.*, 107 (1993), 262–281.
- *C. Bogey, C. Bailly.* A family of low dispersive and low dissipative explicit schemes for flow and noise computations. *J. Comp. Phys.*, 194 (2004), 194–214.

- Very small artificial dissipation and dispersion
- Algorithmic simplicity

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We continue our research from CEAA2012.
Now consider the benchmark of Taylor–Green vortex.

Dispersion-Relation-Preserving schemes

Centered-difference schemes with wide stencils on uniform mesh

$$\left(\frac{\partial\varphi}{\partial x}\right)_j \quad \mapsto \quad \frac{1}{\Delta x}\sum_{l=-m}^{m} a_l \varphi_{j+l}$$

Tam & Webb: $m = 3$ (7-point stencil);

Bogey & Bailly: $m = 4, 5, 6$ (9, 11, 13 points).

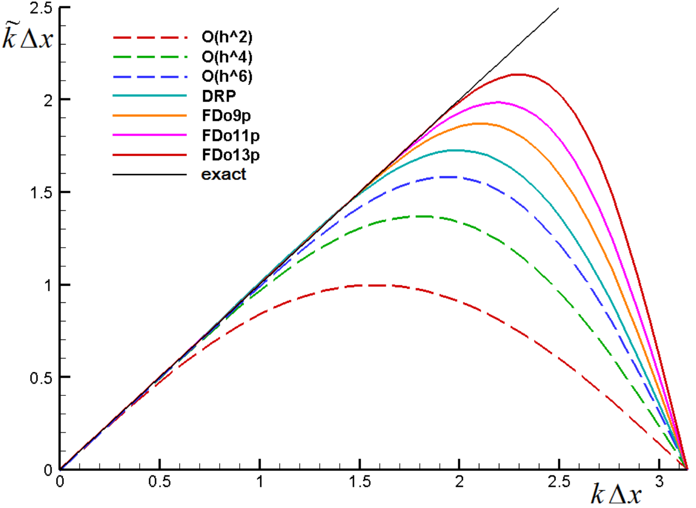
Coefficients a_l are obtained from a special procedure:

4th order in Δx + optimization of spectral resolution.

Effective, or modified, wavenumber

$$\tilde{k}(k) = \frac{2}{\Delta x}\sum_{j=1}^{m} a_j \sin(jk\Delta x)\,.$$ Choose a_j to provide $\tilde{k} \approx k$.

Dispersion relations of various centered-difference schemes



Effective wavenumber $\tilde{k}$ versus true k in various schemes

Application to the Navier–Stokes equations

The 3D Navier–Stokes equations

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}^x}{\partial x} + \frac{\partial \mathbf{F}^y}{\partial y} + \frac{\partial \mathbf{F}^z}{\partial z} = \frac{\partial \mathbf{R}^x}{\partial x} + \frac{\partial \mathbf{R}^y}{\partial y} + \frac{\partial \mathbf{R}^z}{\partial z} \qquad (*)$$

$$\begin{aligned}
\mathbf{Q} &= \begin{pmatrix} \rho & \rho u & \rho v & \rho w & E \end{pmatrix}^T, \\
\mathbf{F}^x &= \begin{pmatrix} \rho u & \rho u^2 + p & \rho u v & \rho u w & (E+p)\, u \end{pmatrix}^T, \\
\mathbf{F}^y &= \begin{pmatrix} \rho v & \rho u v & \rho v^2 + p & \rho v w & (E+p)\, v \end{pmatrix}^T, \\
\mathbf{F}^z &= \begin{pmatrix} \rho w & \rho u w & \rho v w & \rho w^2 + p & (E+p)\, w \end{pmatrix}^T, \\
\mathbf{R}^x &= \begin{pmatrix} 0 & \tau_{xx} & \tau_{xy} & \tau_{xz} & u\tau_{xx} + v\tau_{xy} + w\tau_{xz} - q_x \end{pmatrix}^T, \\
\mathbf{R}^y &= \begin{pmatrix} 0 & \tau_{xy} & \tau_{yy} & \tau_{yz} & u\tau_{xy} + v\tau_{yy} + w\tau_{yz} - q_y \end{pmatrix}^T, \\
\mathbf{R}^z &= \begin{pmatrix} 0 & \tau_{xz} & \tau_{yz} & \tau_{zz} & u\tau_{xz} + v\tau_{yz} + w\tau_{zz} - q_z \end{pmatrix}^T,
\end{aligned}$$

$$\begin{gathered}
\tau_{xx} = \frac{4}{3}\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\frac{\partial w}{\partial z}, \quad \tau_{xy} = \mu\frac{\partial v}{\partial x} + \mu\frac{\partial u}{\partial y}, \\
\ldots, \qquad q_x = -\lambda\frac{\partial T}{\partial x}, \quad q_y = -\lambda\frac{\partial T}{\partial y}, \quad q_z = -\lambda\frac{\partial T}{\partial z}.
\end{gathered} \qquad (**)$$

Application to the Navier–Stokes equations

Numerical approximation to (∗)–(∗∗), 2D case:

$$\frac{d\mathbf{Q}_{jk}}{dt} + \frac{1}{\Delta x}\sum_{l=-m}^{m} a_l \mathbf{F}^x_{j+l,k} + \frac{1}{\Delta y}\sum_{l=-m}^{m} a_l \mathbf{F}^y_{j,k+l}$$

$$= \frac{1}{\Delta x}\sum_{l=-m}^{m} a_l \mathbf{R}^x_{j+l,k} + \frac{1}{\Delta y}\sum_{l=-m}^{m} a_l \mathbf{R}^y_{j,k+l}\,,$$

$$(\tau_{xx})_{jk} = \frac{4}{3}\,\mu\,\frac{1}{\Delta x}\sum_{l=-m}^{m} a_l u_{j+l,k} - \frac{2}{3}\,\mu\,\frac{1}{\Delta y}\sum_{l=-m}^{m} a_l v_{j,k+l}\,, \qquad \text{etc.}$$

Application to the Navier–Stokes equations

Numerical approximation to (∗)–(∗∗), 2D case:

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Numerical filtering

$$\frac{d\mathbf{Q}_{jk}}{dt} + \frac{1}{\Delta x}\sum_{l=-m}^{m} a_l \mathbf{F}^x_{j+l,k} + \frac{1}{\Delta y}\sum_{l=-m}^{m} a_l \mathbf{F}^y_{j,k+l} = \frac{1}{\Delta x}\sum_{l=-m}^{m} a_l \mathbf{R}^x_{j+l,k}$$

$$+ \frac{1}{\Delta y}\sum_{l=-m}^{m} a_l \mathbf{R}^y_{j,k+l} - \frac{\nu_x}{(\Delta x)^2}\sum_{l=-m}^{m} d_l \mathbf{Q}_{j+l,k} - \frac{\nu_y}{(\Delta y)^2}\sum_{l=-m}^{m} d_l \mathbf{Q}_{j,k+l}$$

ν_x, ν_y are empirical filtering rates; d_l are specific filter parameters.

2D Taylor–Green vortex

$$u = U \sin(x/L)\cos(y/L),$$
$$v = -U\cos(x/L)\sin(y/L),$$
$$p = p_0 + (\rho_0 U^2/8)\,[\cos(2x/L) + \cos(2y/L)]\,.$$

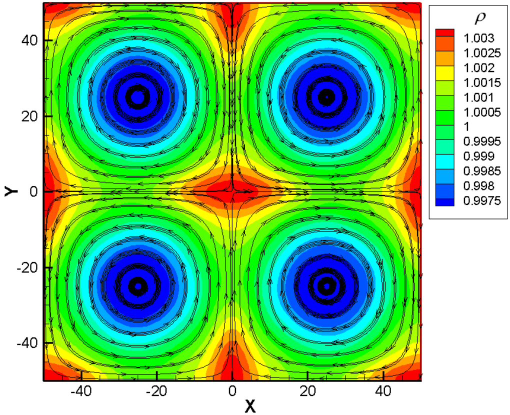
$T = T_0,\ \rho = p/(RT)$

$-\pi L < x, y < \pi L = 50$, periodic boundary conditions in both x and y

$\mathrm{Re} \equiv \rho_0 U L/\mu_0 = 100$,
$M \equiv U/c_0 = 0.1$

Grid $N_x \times N_y = 100 \times 100$,
$\Delta x = \Delta y = 1$

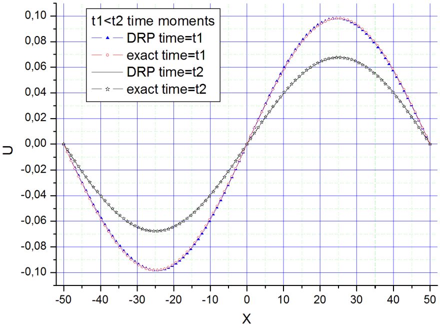
Hereafter, DRP Tam (7-point stencil, $m = 3$)



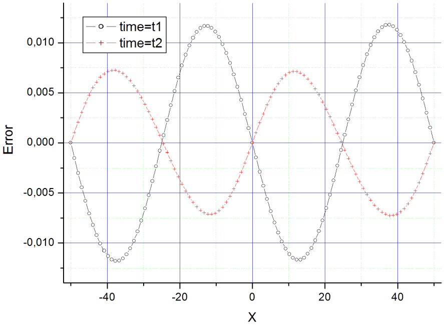
Streamlines and distribution of density ρ

2D Taylor–Green, Re = 100. Comparison with exact solution

$$u_{ex} = u(x, y, 0)\exp\{-2\nu t/L^2\}, \quad v_{ex} = v(x, y, 0)\exp\{-2\nu t/L^2\}.$$



(a)



(b)

(a) Velocity u versus x in section $y = 0$ for two time moments;
(b) error $(u - u_{ex})/U$

3D Taylor–Green vortex

$$u = U\sin(x/L)\cos(y/L)\cos(z/L),$$
$$v = -U\cos(x/L)\sin(y/L)\cos(z/L), \quad w = 0,$$
$$p = p_0 + (\rho_0 U^2/16)\left[\cos(2x/L) + \cos(2y/L)\right](\cos(2z/L) + 2)$$
$$T = T_0, \;\; \rho = p/(RT)$$
$$-\pi L < x, y, z < \pi L = 32,$$

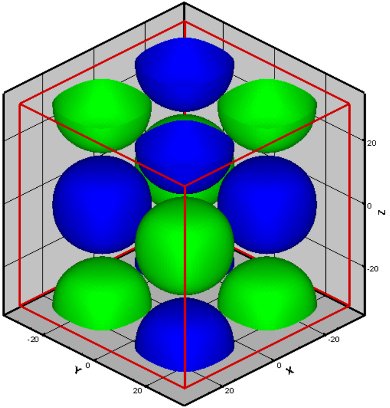
periodic boundary conditions at the 6 faces

Two cases: $\mathrm{Re} = 100$, $\mathrm{Re} = 1600$. $\qquad M = 0.1$

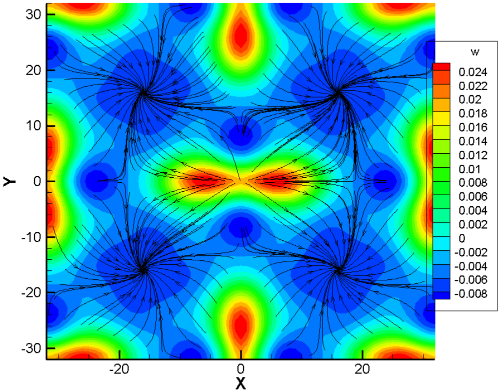
Grid $N_x \times N_y \times N_z = 64^3$, $\Delta x = \Delta y = \Delta z = 1$
DRP Tam

- *Brachet, M., Meiron, D., Orszag, S., et al.* Small-scale structure of the Taylor-Green vortex. *J. Fluid Mech.* (1983) **130**:411–452.
- *Problem C3.5 Direct numerical simulation of the Taylor-Green vortex at Re = 1600.* http://www.as.dlr.de/hiocfd/case_c3.5.pdf

3D Taylor–Green, Re = 100. Flow structure



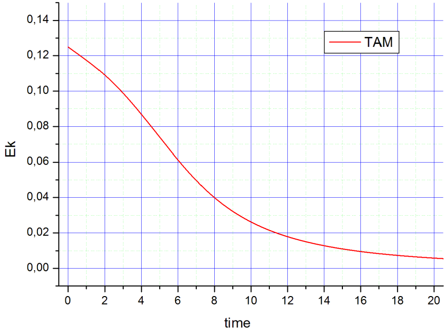
Isosurfaces of z-vorticity $\omega_z = \partial v/\partial x - \partial u/\partial y$ for two values



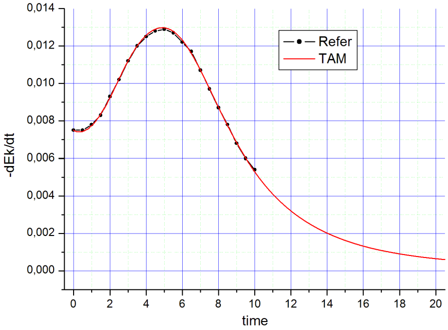
Streamlines and distribution of z-velocity w

3D Taylor–Green, Re = 100. Mean kinetic energy

$$E_k(t) = \frac{1}{8\pi^3 L^3 \rho_0 U^2} \int_\Omega (1/2)\,\rho(u^2 + v^2 + w^2)\, dxdydz$$



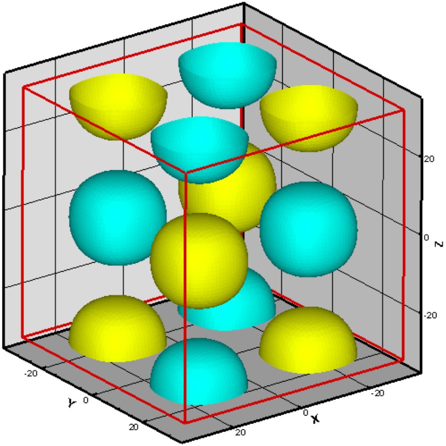
(a)



(b)

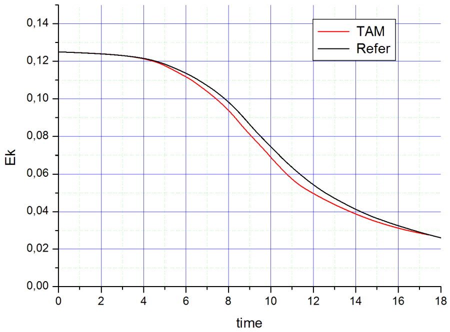
Time evolution of the mean kinetic energy E_k (a) and its dissipation rate $-dE_k/dt$ (b)

3D Taylor–Green, Re = 1600. Flow structure



Isosurfaces of z-vorticity ω_z for two values

3D Taylor–Green, Re = 1600. Mean kinetic energy



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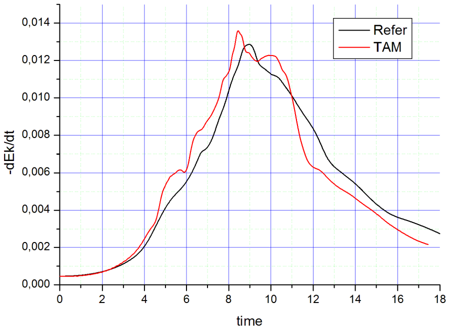
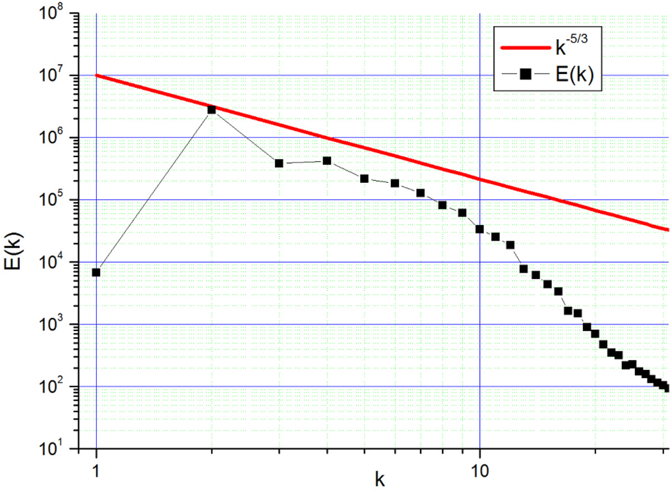


(b)

Time evolution of the mean kinetic energy E_k (a) and its dissipation rate $-dE_k/dt$ (b)

3D Taylor–Green, $\mathrm{Re} = 1600$. Spectrum $E(k)$



Instantaneous energy spectrum of velocity pulsations at $t \approx 9$ compared with the Kolmogorov law $k^{-5/3}$

Conclusion

- DRP technique provides good approximation of the Navier–Stokes viscosity.
- For high (transitional) Reynolds numbers, a filtering procedure is needed to obtain numerical stability.
- Future modifications are expected to enhance quantitative agreement between DRP numerics and the reference data:
  - changing the filter parameters;
  - new approximation of viscous terms?

Acknowledgment
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Thanks for your attention!