Superconvergence and long time simulation accuracy of discontinuous Galerkin method for hyperbolic wave equations

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Introduction

We are interested in solving a hyperbolic conservation law

$$u_t + f(u)_x = 0$$

In 2D it is

$$u_t + f(u)_x + g(u)_y = 0$$

and in system cases u is a vector, and the Jacobian $f'(u)$ is diagonalizable with real eigenvalues.

To solve the hyperbolic conservation law:

$$u_t + f(u)_x = 0, \tag{1}$$

we multiply the equation with a test function v, integrate over a cell $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, and integrate by parts:

$$\int_{I_j} u_t v dx - \int_{I_j} f(u) v_x dx + f(u_{j+\frac{1}{2}}) v_{j+\frac{1}{2}} - f(u_{j-\frac{1}{2}}) v_{j-\frac{1}{2}} = 0$$

Now assume both the solution u and the test function v come from a finite dimensional approximation space V_h, which is usually taken as the space of piecewise polynomials of degree up to k:

$$V_h = \left\{ v : v|_{I_j} \in P^k(I_j),\ j = 1, \cdots, N \right\}$$

However, the boundary terms $f(u_{j+\frac{1}{2}})$, $v_{j+\frac{1}{2}}$ etc. are not well defined when u and v are in this space, as they are discontinuous at the cell interfaces.

From the conservation and stability (upwinding) considerations, we take

- A single valued monotone numerical flux to replace $f(u_{j+\frac{1}{2}})$:

$$\hat{f}_{j+\frac{1}{2}} = \hat{f}(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}})$$

  where $\hat{f}(u,u) = f(u)$ (consistency); $\hat{f}(\uparrow, \downarrow)$ (monotonicity) and $\hat{f}$ is Lipschitz continuous with respect to both arguments.

- Values from inside I_j for the test function v

$$v^-_{j+\frac{1}{2}}, \qquad v^+_{j-\frac{1}{2}}$$

Hence the DG scheme is: find $u \in V_h$ such that

$$\int_{I_j} u_t v dx - \int_{I_j} f(u) v_x dx + \hat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0 \qquad (2)$$

for all $v \in V_h$.

Time discretization could be by the TVD Runge-Kutta method (Shu and Osher, JCP 88). For the semi-discrete scheme:

$$\frac{du}{dt} = L(u)$$

where $L(u)$ is a discretization of the spatial operator, the third order TVD Runge-Kutta is simply:

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n) \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}) \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}) \end{aligned}$$

Properties and advantages of the DG method:

- Easy handling of complicated geometry and boundary conditions (common to all finite element methods). Allowing hanging nodes in the mesh (unique to DG);
- Compact. Communication only with immediate neighbors, regardless of the order of the scheme;
- Explicit. Because of the discontinuous basis, the mass matrix is local to the cell, resulting in explicit time stepping (no systems to solve);
- Parallel efficiency. Achieves 99% parallel efficiency for static mesh and over 80% parallel efficiency for dynamic load balancing with adaptive meshes (Flaherty et al.);

- Provable cell entropy inequality and L^2 stability, for arbitrary nonlinear equations in any spatial dimension and any triangulation, for any polynomial degrees, without limiters or assumption on solution regularity (Jiang and Shu, Math. Comp. 94 (scalar case); Hou and Liu, JSC 07 (symmetric systems)). For $U(u) = \frac{u^2}{2}$:

$$\frac{d}{dt}\int_{I_j} U(u)dx + \hat{F}_{j+1/2} - \hat{F}_{j-1/2} \leq 0$$

Summing over j: $\frac{d}{dt}\int_a^b u^2 dx \leq 0.$

This also holds for fully discrete RKDG methods with third order TVD Runge-Kutta time discretization, for linear equations (Zhang and Shu, SINUM 10).

- At least ($k + \frac{1}{2}$)-th order accurate, and often ($k + 1$)-th order accurate for smooth solutions when piecewise polynomials of degree k are used, regardless of the structure of the meshes, for smooth solutions (Lesaint and Raviart 74; Johnson and Pitkäranta, Math. Comp. 86 (linear steady state); Zhang and Shu, SINUM 04 and 06 (RKDG for nonlinear equations)).

- ($2k + 1$)-th order superconvergence in negative norm and in strong L^2-norm for post-processed solution for linear and nonlinear equations with smooth solutions (Cockburn, Luskin, Shu and Süli, Math. Comp. 03; Ryan, Shu and Atkins, SISC 05; Curtis, Kirby, Ryan and Shu, SISC 07; Ji, Xu and Ryan, JSC in review).

- ($k+3/2$)-th or ($k+2$)-th order superconvergence of the DG solution to a special projection of the exact solution, and non-growth of the error in time up to $t = O(\frac{1}{\sqrt{h}})$ or $t = O(\frac{1}{h})$, for linear and nonlinear hyperbolic and convection diffusion equations (Cheng and Shu, JCP 08; Computers & Structures 09; SINUM 10; Meng, Shu, Zhang and Wu, SINUM to appear (nonlinear); Yang and Shu, SINUM to appear (($k+2$)-th order)).

We consider the DG approximation to smooth solutions of one and two dimensional conservation laws

$$u_t + f(u)_x = b(x,t), \tag{3}$$

and

$$u_t + f(u)_x + g(u)_y = b(x,y,t). \tag{4}$$

We study the convergence and time evolution of the error between the DG solution and the exact solution, as well as the error between the DG solution and a particular projection of the exact solution.

The main conclusion:

- The error between the DG solution and a particular projection of the exact solution superconverges. For P^k elements this error is at least $h^{k+1.5}$ and for some cases it can be h^{k+2} (half or one order higher than usual).

- As a consequence, the error between the DG solution and the exact solution does not grow with time, for a long time period $0 \le t \le O(h^{-0.5})$ or $0 \le t \le O(h^{-1})$.

Related work:

- Adjerid et al. (CMAME 2002, 2006) proved superconvergence of the DG solutions at Radau points for ordinary differential equations. They have also made numerical experiments for the partial differential equations.

- Zhang and Shu (Computers & Fluids 2005) explicitly give the formulation of the DG solution in the case of P^1 (piecewise linear) for the linear convection equation. The leading error term is shown to be of a constant magnitude independent of the time t.

Linear equations with constant coefficients

We consider the following equation

$$\begin{cases} u_t + u_x = 0 \\ u(x,0) = u_0(x) \\ u(0,t) = u(2\pi,t) \end{cases} . \tag{5}$$

Here, $u_0(x)$ is a smooth 2π-periodic function.

The assumption of periodic boundary condition is not essential, the same results hold also for initial-boundary value problems.

We define $P_h^- u$ to be a projection of u into V_h^k, such that

$$\int_{I_j} P_h^- u \, v_h \, dx = \int_{I_j} u \, v_h \, dx \tag{6}$$

for any $v_h \in P^{k-1}$ on I_j, where k is the polynomial degree of the DG solution, and

$$(P_h^- u)^- = u^- \qquad \text{at } x_{j+1/2}. \tag{7}$$

Note: this special projection is used in the error estimates of the DG methods to derive optimal L^2 error bounds in the literature.

We are going to show that indeed the numerical solution is closer to this special projection of the exact solution than to the exact solution itself.

Let us denote:

- $e = u - u_h$ to be the error between the exact solution and numerical solution
- $\varepsilon = u - P_h^- u$ to be the projection error
- $\bar{e} = P_h^- u - u_h$ to be the error between the numerical solution and the projection of the exact solution.

Theorem: Let u be the exact solution of the equation (5), and u_h be the DG solution with suitable initial condition. We have the following error estimate:

$$||\bar{e}(\cdot,t)||_{L^2} \leq C_1\,(t+1)\,h^{k+2}, \tag{8}$$

and

$$||e(\cdot,t)||_{L^2} \leq C_1\,t\,h^{k+2} + C_2\,h^{k+1}, \tag{9}$$

where C_1 and C_2 are constants which do not depend on t or h.

Even though the theorem is stated for the simple scalar equation, the same proof applies also to any linear hyperbolic system

$$u_t + Au_x = 0$$

where A is a constant matrix which is diagonalizable with real eigenvalues. This is because the PDE as well as the DG scheme can be diagonalized into decoupled scalar equations.

Example 1. We solve the one dimensional equation

$$\begin{cases} u_t + u_x = 0 \\ u(x,0) = \sin(x) \\ u(0,t) = u(2\pi,t) \end{cases} . \tag{10}$$

The errors $\bar{e}$ and e using P^1 polynomials on a uniform mesh of N cells are shown in Table 1. Notice that $\bar{e}$ is superconvergent of order 3 and grows linearly with t, while e is of order 2 but does not grow with time until very large t.

Table 1: Example 1, P^1 polynomials, uniform mesh

		$T=1$		$T=10$		$T=100$	
	N	L^2 error	order	L^2 error	order	L^2error	order
$\bar{e}$	20	4.60E-04	-	3.04E-03	-	2.96E-02	-
	40	5.80E-05	2.99	3.82E-04	2.99	3.79E-03	2.97
	80	7.26E-06	3.00	4.79E-05	3.00	4.75E-04	2.99
	160	9.08E-07	3.00	5.99E-06	3.00	5.95E-05	3.00
e	20	4.21E-03	-	5.16E-03	-	2.99E-02	-
	40	1.06E-03	1.99	1.12E-03	2.20	3.93E-03	2.92
	80	2.65E-04	2.00	2.69E-04	2.06	5.44E-04	2.85
	160	6.64E-05	2.00	6.66E-05	2.02	8.91E-05	2.61

The same result holds for the initial-boundary problem

$$\begin{cases} u_t + u_x = 0 \\ u(x,0) = \sin(x) \\ u(0,t) = \sin(-t) \end{cases} . \tag{11}$$

See Table 2.

Table 2: Initial-boundary value problem (11), P^1, uniform meshes.

		$T=1$		$T=10$		$T=100$	
	N	L^2 error	order	L^2 error	order	L^2error	order
$\bar{e}$	20	4.74E-04	-	1.28E-03	-	1.06E-03	-
	40	6.02E-05	2.98	1.60E-04	3.00	1.33E-04	3.00
	80	7.57E-06	2.99	2.00E-05	3.00	1.65E-05	3.01
	160	9.50E-07	3.00	2.50E-06	3.00	2.16E-06	2.93
e	20	4.22E-03	-	4.43E-03	-	4.37E-03	-
	40	1.06E-03	1.99	1.07E-03	2.04	1.07E-03	2.03
	80	2.65E-04	2.00	2.66E-04	2.01	2.66E-04	2.01
	160	6.64E-05	2.00	6.64E-05	2.00	6.64E-05	2.00

Example 2. We still solve the same equation as in the previous example but with a different initial condition.

$$\begin{cases} u_t + u_x = 0 \\ u(x,0) = e^{\sin(x)} \\ u(0,t) = u(2\pi,t) \end{cases} \tag{12}$$

The initial condition is no longer a single mode. The result is the same as before, in Table 3 for a random mesh.

Table 3: Example 2. P^1 polynomials, random meshes.

		$T=1$		$T=10$		$T=100$	
	N	L^2 error	order	L^2 error	order	L^2error	order
$\bar{e}$	20	2.05E-03	-	1.66E-02	-	1.08E-01	-
	40	2.80E-04	2.87	2.32E-03	2.84	2.07E-02	2.38
	80	3.43E-05	3.03	2.89E-04	3.01	2.83E-03	2.87
	160	4.53E-06	2.92	3.67E-05	2.98	3.66E-04	2.95
e	20	6.79E-03	-	1.78E-02	-	1.08E-01	-
	40	1.79E-03	1.92	2.88E-03	2.63	2.08E-02	2.37
	80	4.31E-04	2.05	5.28E-04	2.45	2.87E-03	2.86
	160	1.11E-04	1.95	1.17E-04	2.18	3.82E-04	2.91

Example 2a. We solve the same problem

$$\begin{cases} u_t + u_x = 0 \\ u(x,0) = e^{\sin(x)} \\ u(0,t) = u(2\pi,t) \end{cases}$$

using P^2 and P^3. Similar results hold as before, in Tables 4 and 5 for uniform meshes.

Table 4: Example 2a. P^2 polynomials, uniform meshes.

		$T=1$		$T=100$		$T=1000$	
	N	L^2 error	order	L^2 error	order	L^2error	order
$\bar{e}$	20	2.64E-05	-	1.22E-03	-	1.01E-02	-
	40	1.59E-06	4.05	4.07E-05	4.90	4.02E-04	4.64
	80	9.77E-08	4.03	1.29E-06	4.98	1.28E-05	4.97
	160	6.08E-09	4.01	4.07E-08	4.98	4.02E-07	5.00
e	20	2.94E-04	-	1.25E-03	-	1.01E-02	-
	40	3.67E-05	3.00	5.48E-05	4.51	4.04E-04	4.64
	80	4.59E-06	3.00	4.77E-06	3.52	1.36E-05	4.89
	160	5.74E-07	3.00	5.76E-07	3.05	7.01E-07	4.28

Table 5: Example 2a. P^3 polynomials, uniform meshes.

		$T=10$		$T=100$		$T=500$	
	N	L^2 error	order	L^2 error	order	L^2error	order
$\bar{e}$	5	5.40E-03	-	2.86E-02	-	6.70E-02	-
	10	8.70E-05	5.96	7.65E-04	5.22	3.24E-03	4.37
	20	1.11E-06	6.30	7.50E-06	6.67	3.71E-05	6.45
	40	2.61E-08	5.40	6.57E-08	6.84	3.04E-07	6.93
e	5	5.50E-03	-	2.89E-02	-	6.70E-02	-
	10	2.09E-04	4.72	7.88E-04	5.19	3.24E-03	4.37
	20	1.22E-05	4.09	1.43E-05	5.79	3.91E-05	6.38
	40	7.65E-07	4.00	7.67E-07	4.22	8.22E-07	5.57

Finally we consider the case of P^0. In this case, the projection P_h^- can no longer be defined. We compute e for Example 1 when $N = 320$ and list the L^2 errors in Table 6. Unlike the cases of P^1, P^2, and P^3, this time e grows with respect to time even for fine grids.

Table 6: The error e for Example 1. P^0 polynomials, uniform meshes of 320 cells.

T	L^2 error
1	7.99E-03
10	6.62E-02
100	4.42E-01

Linear equations with variable coefficients

Example 3. We solve the following equation

$$\begin{cases} u_t + (a(x)u)_x = b(x,t) \\ u(x,0) = \sin(x) \\ u(0,t) = u(2\pi,t) \end{cases}, \tag{13}$$

where $a(x)$ and $b(x,t)$ are given by

$$a(x) = \sin(x) + 2,$$
$$b(x,t) = (\sin(x)+3)\cos(x+t) + \cos(x)\sin(x+t).$$

Notice that $a(x) > 0$, we can still use the upwind fluxes. Similar (or even better) results are observed, in Table 7.

Table 7: Example 3. P^2 polynomials, uniform meshes.

		$T = 1$		$T = 100$		$T = 500$	
	N	L^2 error	order	L^2 error	order	L^2error	order
$\bar{e}$	20	4.29E-06	-	4.19E-06	-	4.20E-06	-
	40	2.64E-07	4.02	2.61E-07	4.00	2.62E-07	4.00
	80	1.65E-08	4.00	1.63E-08	4.00	1.63E-08	4.00
	160	1.03E-09	4.00	1.02E-09	4.00	1.02E-09	4.00
e	20	1.07E-04	-	1.07E-04	-	1.07E-04	-
	40	1.34E-05	3.00	1.34E-05	3.00	1.34E-05	3.00
	80	1.67E-06	3.00	1.67E-06	3.00	1.67E-06	3.00
	160	2.09E-07	3.00	2.09E-07	3.00	2.09E-07	3.00

Example 4. We solve the same equation

$$\begin{cases} u_t + (a(x)u)_x = b(x,t) \\ u(x,0) = \sin(x) \\ u(0,t) = u(2\pi,t) \end{cases},$$

where $a(x)$ and $b(x,t)$ are given by

$$a(x) = \sin(x),$$
$$b(x,t) = (\sin(x)+1)\cos(x+t) + \cos(x)\sin(x+t).$$

This time $a(x)$ is no longer always positive, but we still use the upwind flux.

The projection P_h is defined as follows. If $a(x_j) > 0$, then on the cell I_j, we use P_h^-; otherwise, we use P_h^+, which is defined as the projection of u into V_h^k such that

$$\int_{I_j} P_h^+ u \, v_h \, dx = \int_{I_j} u \, v_h \, dx$$

for any $v_h \in P^{k-1}$ on I_j and

$$(P_h^+ u)^+ = u^+ \qquad \text{at } x_{j-1/2}.$$

Similar results are observed as before in Table 8.

Table 8: Example 4. P^2 polynomials, uniform meshes.

		$T = 1$		$T = 100$		$T = 500$	
	N	L^2 error	order	L^2 error	order	L^2error	order
$\bar{e}$	20	4.38E-05	-	9.73E-05	-	8.53E-05	-
	40	3.96E-06	3.47	5.99E-06	4.02	9.30E-06	3.20
	80	3.53E-07	3.49	4.11E-07	3.87	5.26E-07	4.14
	160	3.13E-08	3.50	3.93E-08	3.38	4.12E-08	4.00
e	20	1.16E-04	-	1.27E-04	-	1.27E-04	-
	40	1.40E-05	3.05	1.41E-05	3.17	1.54E-05	3.05
	80	1.72E-06	3.03	1.75E-06	3.01	1.76E-06	3.13
	160	2.12E-07	3.02	2.13E-07	3.04	2.16E-07	3.00

Nonlinear equations

Example 5. We solve the following nonlinear equation

$$\begin{cases} u_t + (u^3)_x = b(x,t) \\ u(x,0) = \sin(x) \\ u(0,t) = u(2\pi,t) \end{cases}, \tag{14}$$

where $b(x,t)$ is given by

$$b(x,t) = (1 + 3\sin^2(x+t))\cos(x+t).$$

Since $f'(u) = 3u^2 \geq 0$, we can still use the upwind fluxes. Similar results are achieved, in Table 9.

Table 9: Example 5. P^2 polynomials, uniform meshes.

		$T=1$		$T=100$		$T=500$	
	N	L^2 error	order	L^2 error	order	L^2error	order
$\bar{e}$	20	5.40E-05	-	3.45E-05	-	3.53E-05	-
	40	4.67E-06	3.53	3.02E-06	3.51	3.01E-06	3.55
	80	3.22E-07	3.86	2.57E-07	3.56	2.57E-07	3.55
	160	1.99E-08	4.02	1.91E-08	3.75	1.91E-08	3.75
e	20	1.12E-04	-	1.08E-04	-	1.08E-04	-
	40	1.34E-05	3.07	1.33E-05	3.03	1.33E-05	3.03
	80	1.65E-06	3.02	1.65E-06	3.01	1.65E-06	3.01
	160	2.07E-07	3.00	2.07E-07	2.99	2.07E-07	2.99

Example 6. We solve the following nonlinear Burgers equation

$$\begin{cases} u_t + (u^2)_x = b(x,t) \\ u(x,0) = \sin(x) \\ u(0,t) = u(2\pi,t) \end{cases}, \tag{15}$$

where $b(x,t)$ is given by

$$b(x,t) = (1 + 2\sin(x+t))\cos(x+t).$$

Now, $f'(u)$ is no longer always positive. We use the Godunov flux, which is an upwind flux. The projection P_h is defined as follows. If $u(x_j,t)$ is positive, then on the cell I_j, we use P_h^-; otherwise, we use P_h^+. See Table 10 for similar results as before.

Table 10: Example 6. P^2 polynomials, uniform meshes.

	N	$T=1$ L^2 error	$T=1$ order	$T=100$ L^2 error	$T=100$ order	$T=500$ L^2 error	$T=500$ order
$\bar{e}$	20	7.18E-05	-	8.97E-05	-	1.14E-04	-
	40	7.56E-06	3.53	9.58E-06	3.23	9.03E-06	3.66
	80	9.19E-07	3.04	8.84E-07	3.44	8.92E-07	3.34
	160	7.76E-08	3.57	7.71E-08	3.51	7.84E-08	3.51
e	20	1.23E-04	-	1.37E-04	-	1.36E-04	-
	40	1.48E-05	3.05	1.53E-05	3.16	1.54E-05	3.14
	80	1.79E-06	3.05	1.81E-06	3.08	1.81E-06	3.09
	160	2.16E-07	3.05	2.16E-07	3.00	2.16E-07	3.07

One-dimensional systems

Example 7. We solve the following one dimensional system

$$\begin{cases} \begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ u(x,0) = \sin(x) \\ v(x,0) = \cos(x) \\ u(0,t) = u(2\pi,t) \\ v(0,t) = v(2\pi,t) \end{cases} . \qquad (16)$$

We take the upwind flux and define the projection P_h according to the wind directions. The results are similar to the scalar cases, in Table 11.

Table 11: Example 7. P^2 polynomials, uniform meshes.

		$T=1$		$T=10$		$T=100$	
	N	L^2 error	order	L^2 error	order	L^2error	order
$\bar{e}$	20	4.17E-06	-	5.13E-06	-	3.02E-05	-
	40	2.62E-07	3.99	2.78E-07	4.21	9.74E-07	4.95
	80	1.64E-08	4.00	1.66E-08	4.06	3.36E-08	4.86
	160	1.02E-09	4.00	1.03E-09	4.02	1.37E-09	4.61
e	20	1.07E-04	-	1.07E-04	-	1.11E-04	-
	40	1.34E-05	3.00	1.34E-05	3.00	1.34E-05	3.05
	80	1.67E-06	3.00	1.67E-06	3.00	1.67E-06	3.00
	160	2.09E-07	3.00	2.09E-07	3.00	2.09E-07	3.00

Two-dimensional equations

Example 8. We solve the following equation

$$\begin{cases} u_t + u_x + u_y = 0 \\ u(x, y, 0) = \sin(x + y) \end{cases} . \tag{17}$$

Periodic boundary conditions are imposed on the boundary of the domain $[0, 2\pi]^2$. We use rectangular meshes and Q^k elements. The results are similar to the 1D case, in Table 12.

Table 12: Example 8. Q^1 elements, uniform meshes.

		$T=1$		$T=10$		$T=100$	
	N	L^2 error	order	L^2 error	order	L^2error	order
$\bar{e}$	5	4.68E-02	-	2.81E-01	-	7.01E-01	-
	10	6.64E-03	2.82	4.60E-02	2.61	3.46E-01	1.02
	20	8.63E-04	2.94	6.04E-03	2.93	5.79E-02	2.58
	40	1.09E-04	2.98	7.63E-04	2.99	7.56E-03	2.94
e	5	8.80E-02	-	2.86E-01	-	7.06E-01	-
	10	2.34E-02	1.91	5.06E-02	2.50	3.47E-01	1.03
	20	5.97E-03	1.97	8.42E-03	2.59	5.82E-02	2.57
	40	1.50E-03	1.99	1.68E-03	2.33	7.70E-03	2.92

We can extend the result to the linear convection-diffusion equation

$$\begin{cases} u_t + cu_x = bu_{xx} \\ u(x,0) = u_0(x) \\ u(0,t) = u(2\pi,t). \end{cases} \tag{18}$$

The LDG scheme for (18) uses the same mesh and approximation space as in the hyperbolic case and is formulated based on rewriting (18) into

$$\begin{cases} u_t + cu_x = aq_x \\ q - au_x = 0. \end{cases} \tag{19}$$

Here $a = \sqrt{b}$.

The scheme is, find $u_h, q_h \in V_h^k$, such that

$$\int_{I_j} (u_h)_t v_h dx - \int_{I_j} c u_h (v_h)_x dx + c\tilde{u}_h v_h^- |_{j+\frac{1}{2}} - c\tilde{u}_h v_h^+ |_{j-\frac{1}{2}} + \int_{I_j} a q_h (v_h)_x dx - a\hat{q}_h v_h^- |_{j+\frac{1}{2}} + a\hat{q}_h v_h^+ |_{j-\frac{1}{2}} = 0, \quad (20)$$

$$\int_{I_j} q_h w_h dx + \int_{I_j} a u_h (w_h)_x dx - a\hat{u}_h w_h^- |_{j+\frac{1}{2}} + a\hat{u}_h w_h^+ |_{j-\frac{1}{2}} = 0$$

hold for any $v_h, w_h \in V_h^k$, where $\tilde{u}_h$ is the upwind flux depending on the sign of c. Without loss of generality we assume $c \geq 0$ and $\tilde{u}_h = u_h^-$. The alternating diffusion fluxes are taken as

$$\hat{q}_h = q_h^+, \qquad \hat{u}_h = u_h^-. \quad (21)$$

Theorem: If $k \geq 1$, let u, $q = u_x$ be the exact solution of the convection diffusion equation (18) when $c > 0$ and $\tilde{u}_h = u_h^-$, and u_h, q_h be the LDG solution. We define $P_h u = P_h^- u$, $P_h q = P_h^+ q$, and we choose the initial condition as $u_h(\cdot, 0) = P_h^1 u_0$. Then we have the following error estimate:

$$||\bar{e}_u(\cdot, t)||_{L^2}^2 + \int_0^t ||\bar{e}_q(\cdot, s)||_{L^2}^2 \, ds \leq C h^{2k+3} (t+1)^2,$$

and in particular

$$||\bar{e}_u(\cdot, t)||_{L^2} \leq C h^{k+3/2} (t+1).$$

where $C = C(||u||_{k+5}, \lambda, a/c)$.

Example 9. We solve the heat equation

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = \sin(x) \\ u(0,t) = u(2\pi,t). \end{cases}$$

Table 13: Example 9, P^0 polynomials, uniform mesh, $T = 1$.

	e_u		$e_u(x_j)$	
N	L^2 error	order	L^2error	order
10	4.76E-02	-	4.19E-03	-
20	2.36E-02	1.01	1.06E-03	1.98
40	1.18E-02	1.00	2.67E-04	1.99
80	5.90E-03	1.00	6.68E-05	2.00
	e_q		$e_q^+(x_{j-1/2})$	
10	1.55E-01	-	1.20E-03	-
20	4.71E-02	0.99	1.93E-04	2.64
40	2.36E-02	1.00	8.22E-06	4.55
80	1.02E-02	1.00	1.02E-07	6.33

Table 14: Example 9, P^1 polynomials, uniform mesh, $T = 1$.

	e_u		$\bar{e}_u$		$e_u^-(x_{j+1/2})$	
N	L^2 error	order	L^2 error	order	L^2error	order
10	6.28E-03	-	1.04E-03	-	9.82E-05	-
20	1.56E-03	2.01	1.29E-04	3.00	5.93E-06	4.05
40	3.91E-04	2.00	1.62E-05	3.00	3.68E-07	4.01
80	9.77E-05	2.00	2.02E-06	3.00	2.29E-08	4.00

	e_q		$\bar{e}_q$		$e_q^+(x_{j-1/2})$	
10	6.35E-03	-	5.32E-04	-	9.22E-04	-
20	1.57E-03	2.02	6.51E-05	3.03	1.13E-04	3.03
40	3.91E-04	2.00	8.10E-06	3.01	1.40E-05	3.01
80	9.77E-05	2.00	1.01E-06	3.00	1.75E-06	3.00

Table 15: Example 9, P^2 polynomials, uniform mesh, $T = 1$.

	e_u		$\bar{e}_u$		$e_u^-(x_{j+1/2})$	
N	L^2 error	order	L^2 error	order	L^2error	order
10	3.15E-04	-	2.53E-05	-	3.62E-06	-
20	3.94E-05	3.00	1.55E-06	4.03	1.11E-07	5.03
40	4.92E-06	3.00	9.65E-08	4.01	3.46E-09	5.01
80	6.15E-07	3.00	6.03E-09	4.00	1.08E-10	5.00
	e_q		$\bar{e}_q$		$e_q^+(x_{j-1/2})$	
10	3.15E-04	-	2.51E-05	-	4.73E-07	-
20	3.94E-05	3.00	1.55E-06	4.02	8.29E-09	5.84
40	4.92E-06	3.00	9.65E-08	4.00	8.90E-11	6.54
80	6.15E-07	3.00	6.03E-09	4.00	1.17E-12	6.25

Table 16: Example 9, P^3 polynomials, uniform mesh, $T = 1$.

	e_u		$\bar{e}_u$		$e_u^-(x_{j+1/2})$	
N	L^2 error	order	L^2 error	order	L^2error	order
5	1.93E-04	-	2.08E-05	-	1.78E-07	-
10	1.21E-05	3.99	6.37E-07	5.03	5.15E-10	8.43
20	7.60E-07	4.00	1.98E-08	5.01	2.13E-12	7.92
	e_q		$\bar{e}_q$		$e_q^+(x_{j-1/2})$	
5	1.93E-04	-	2.08E-05	-	1.02E-06	-
10	1.21E-05	3.99	6.37E-07	5.03	7.31E-09	7.13
20	7.60E-07	4.00	1.98E-08	5.01	5.60E-11	7.03

Our numerical results show that all the aforementioned errors decay exponentially with respect to time, see Figure 1.

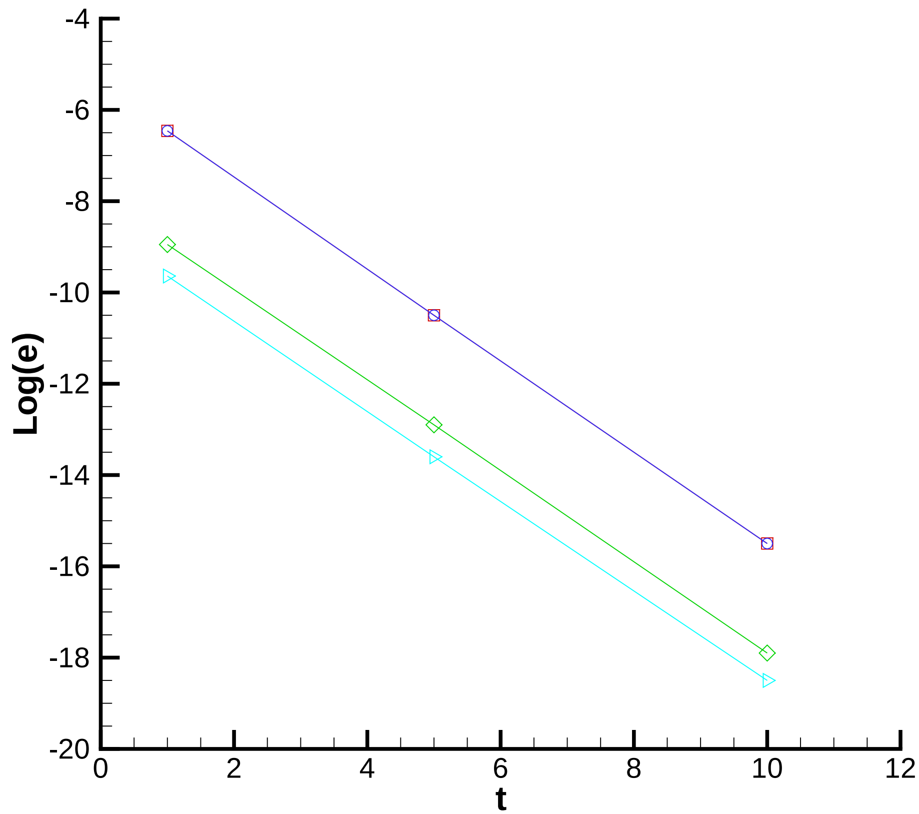

Figure 1: Example 9, errors versus time when using P^1 polynomials on a uniform mesh of 20 cells. Squares: e_u; Diamonds: $\bar{e}_u$; Circles: e_q; Right Triangles: $\bar{e}_q$.

As expected by theory, the superconvergence result holds also for non-uniform meshes. In the next table, we list the errors and orders for a non-uniform mesh which is a 10% random perturbation of the uniform mesh. We can see that all the conclusions for uniform meshes also hold true for this non-uniform mesh.

Table 17: Example 9, P^1 polynomials, random mesh, $T = 1$.

	e_u		$\bar{e}_u$		$e_u^-(x_{j+1/2})$	
N	L^2 error	order	L^2 error	order	L^2error	order
10	6.94E-03	-	1.02E-03	-	1.81E-04	-
20	1.61E-03	2.10	1.43E-04	2.84	9.52E-06	4.25
40	4.07E-04	1.99	1.76E-05	3.02	6.53E-07	3.87
80	1.09E-04	1.90	2.30E-06	2.94	4.20E-08	3.96

	e_q		$\bar{e}_q$		$e_q^+(x_{j-1/2})$	
10	6.15E-03	-	6.97E-04	-	1.05E-03	-
20	1.64E-03	1.90	7.66E-05	3.19	1.18E-04	3.15
40	4.07E-04	2.01	9.57E-06	3.00	1.47E-05	3.00
80	1.09E-04	1.90	1.23E-06	2.96	1.97E-06	2.90

Example 10. We solve the convection-dominated problem

$$\begin{cases} u_t + u_x = 0.01 u_{xx} \\ u(x,0) = \sin(x) \\ u(0,t) = u(2\pi, t). \end{cases}$$

$(k+2)$-th order superconvergence are obtained for $\bar{e}_u$, $\bar{e}_q$ and the point errors u^- and q^+. The only difference is that we now need a more refined mesh to observe superconvergence for $\bar{e}_q$.

Table 18: Example 10, P^1 polynomials, uniform mesh. $T = 1$.

	e_u		$\bar{e}_u$		$e_u^-(x_{j+1/2})$	
N	L^2 error	order	L^2 error	order	L^2error	order
10	1.64E-02	-	3.47E-03	-	4.78E-03	-
20	4.18E-03	1.97	4.37E-04	2.99	5.82E-04	3.04
40	1.05E-03	1.99	5.38E-05	3.02	6.61E-05	3.14
80	2.63E-04	2.00	6.63E-06	3.02	7.13E-06	3.21
	e_q		$\bar{e}_q$		$e_q^+(x_{j-1/2})$	
160	4.17E-06	-	4.11E-06	-	7.06E-06	-
320	1.20E-06	1.79	6.40E-07	2.68	1.10E-06	2.68
640	3.43E-07	1.81	9.13E-08	2.81	1.57E-07	2.81
1280	9.33E-08	1.88	1.23E-08	2.89	2.11E-08	2.89

Table 19: Example 10, P^2 polynomials, uniform mesh. $T = 1$.

	e_u		$\bar{e}_u$		$e_u^-(x_{j+1/2})$	
N	L^2 error	order	L^2 error	order	L^2error	order
10	8.43E-04	-	7.16E-05	-	3.12E-05	-
20	3.94E-05	3.00	1.40E-06	3.89	1.48E-07	4.92
40	4.92E-06	3.00	9.17E-08	3.93	4.75E-09	4.96
80	6.15E-07	3.00	5.87E-09	3.96	1.51E-10	4.98
	e_q		$\bar{e}_q$		$e_q^+(x_{j-1/2})$	
160	1.68E-08	-	1.58E-08	-	1.10E-08	-
320	2.18E-09	2.95	1.14E-09	3.79	4.50E-10	4.61
640	2.92E-10	2.90	7.46E-11	3.93	1.58E-11	4.83
1280	3.83E-11	2.93	2.97E-12	4.65	4.48E-13	5.14

Table 20: Example 10, P^3 polynomials, uniform mesh. $T = 1$.

	e_u		$\bar{e}_u$		$e_u^-(x_{j+1/2})$	
N	L^2 error	order	L^2 error	order	L^2error	order
5	5.24E-04	-	7.53E-05	-	9.75E-05	-
10	3.36E-05	3.96	2.48E-06	4.93	1.88E-06	5.70
20	2.05E-06	4.04	4.47E-08	5.79	4.97E-08	5.24
40	1.28E-07	3.98	1.24E-09	5.17	1.86E-09	4.74

	e_q		$\bar{e}_q$		$e_q^+(x_{j-1/2})$	
80	6.19E-10	-	6.12E-10	-	5.07E-10	-
160	4.07E-11	3.93	2.59E-11	4.56	8.31E-12	5.93
320	2.74E-12	3.89	9.29E-13	4.80	9.00E-14	6.53

Example 11. We solve the following equation

$$\begin{cases} u_t + (u^2)_x = (b(u)u_x)_x + c(x,t) \\ u(x,0) = \sin(x) \\ u(0,t) = u(2\pi,t) \end{cases}$$

with

$$\begin{aligned} b(u) &= u^4, \\ c(x,t) &= -e^{-5t}(e^{4t}\cos(t-x) - e^{4t}\sin(t-x) \\ &\quad -4\cos^2(t-x)\sin^3(t-x) + \sin^5(t-x) + e^{3t}\sin(2(t-x))) \end{aligned}$$

This is a convection diffusion equation which is nonlinear in both convection and diffusion, and the wind direction for the convection term changes sign.

We use the Godunov flux for the convection term, and the alternating fluxes for the diffusion fluxes. The numerical traces are $w_1 = \widehat{u_h^2} - \frac{[g(u_h)]}{[u_h]} q_h^+$ and $w_2 = g(u_h^-)$. The errors for w_1 and w_2 are $e_{w_1} = w_1 - u^2 + a(u)q = w_1 - u^2 + u^4 u_x$ and $e_{w_2} = w_2 - g(u) = w_2 - u^3/3$. The results are listed in the next table. We clearly observe $(k+2)$-th order superconvergence for both numerical traces.

Table 21: Example 11 on a uniform mesh of N cells. $T = 1$.

		e_u		e_q		$e_{w_1}(x_{j+1/2})$		$e_{w_2}(x_{j+1/2})$	
P^1	N	L^2 error	order	L^2 error	order	L^2err or	order	L^2error	order
	20	1.47E-03	-	2.50E-03	-	2.96E-04	-	2.31E-04	-
	40	3.42E-04	2.10	9.96E-04	1.33	3.74E-05	2.98	4.51E-05	2.35
	80	8.36E-05	2.03	3.58E-04	1.48	5.04E-06	2.89	8.08E-06	2.48
P^2	N	L^2 error	order	L^2 error	order	L^2error	order	L^2error	order
	10	5.26E-04	-	3.24E-04	-	2.33E-04	-	6.10E-05	-
	20	5.99E-05	3.13	5.71E-05	2.51	2.50E-05	3.22	5.57E-06	3.45
	40	6.05E-06	3.31	9.60E-06	2.57	1.28E-06	4.29	3.77E-07	3.88
	80	6.73E-07	3.17	1.68E-06	2.52	7.03E-08	4.18	2.87E-08	3.71
P^3	N	L^2 error	order	L^2 error	order	L^2error	order	L^2error	order
	5	4.07E-04	-	3.78E-04	-	1.23E-04	-	3.15E-05	-
	10	1.61E-05	4.66	3.18E-05	3.57	4.59E-06	4.74	2.39E-06	3.72
	20	7.42E-07	4.44	2.51E-06	3.66	7.80E-08	5.88	8.38E-08	4.84
	40	4.43E-08	4.07	1.99E-07	3.66	2.50E-09	4.96	2.63E-09	4.99

Conclusion and future work

- The DG solution is superconvergent when measured by its distance to a special projection of the exact solution.
- As a consequence, the usual L^2 error of the DG solution does not grow with time for a "long" time, typically $0 \leq t \leq O(h^{-1})$ and at least $0 \leq t \leq O(h^{-0.5})$.
- The conclusion holds true for very general cases: non-uniform meshes, variable coefficient PDEs, nonlinear PDEs, systems, 2D, with possible change of wind directions, and also for other norms measuring the error (L^1, L^∞, etc.). The proof is given for linear and certain nonlinear PDEs in L^2 norm.

- References:

  1. Y. Cheng and C.-W. Shu, *Superconvergence and time evolution of discontinuous Galerkin finite element solutions*, Journal of Computational Physics, v227 (2008), pp.9612-9627.

     We use Fourier analysis to prove the superconvergence of order $O(h^{\frac{5}{2}})$ for P^1 uniform mesh cases for one dimensional linear hyperbolic equation with constant coefficients, and provide numerical results for many general cases.

2. Y. Cheng and C.-W. Shu, *Superconvergence of local discontinuous Galerkin methods for one-dimensional convection-diffusion equations*, Computers & Structures, v87 (2009), pp.630-641.

   We use Fourier analysis to prove the superconvergence of order $O(h^{\frac{5}{2}})$ for P^1 uniform mesh cases for one dimensional linear convection diffusion equation with constant coefficients.

3. Y. Cheng and C.-W. Shu, *Superconvergence of discontinuous Galerkin and local discontinuous Galerkin schemes for linear hyperbolic and convection diffusion equations in one space dimension*, SIAM Journal on Numerical Analysis, v47 (2010), pp.4044-4072.

   We use a finite element technique to prove the superconvergence of order $O(h^{k+\frac{3}{2}})$ for general P^k cases with arbitrary meshes for one dimensional linear hyperbolic and convection diffusion equation with constant coefficients.

4. Y. Yang and C.-W. Shu, *Analysis of optimal superconvergence of discontinuous Galerkin method for linear hyperbolic equations*, SIAM Journal on Numerical Analysis, to appear.

   We use a new technique to prove the superconvergence of order $O(h^{k+2})$ for general P^k cases with arbitrary meshes for one dimensional linear hyperbolic equation with constant coefficients.

5. X. Meng, C.-W. Shu, Q. Zhang and B. Wu, *Superconvergence of discontinuous Galerkin method for scalar nonlinear conservation laws in one space dimension*, SIAM Journal on Numerical Analysis, to appear.

   We generalize the proof to the superconvergence of order $O(h^{k+\frac{3}{2}})$ for general P^k cases with arbitrary meshes for one dimensional scalar nonlinear hyperbolic conservation laws.

- Future work:
  - Proof for more general cases.
  - More general PDEs: KdV, ...
  - Applications: use it in adaptive computation.

The End

THANK YOU!