A COMBINED FEM/BEM DISCRETE NUMERICAL METHOD FOR SOLVING EXTERNAL SCATTERING PROBLEMS IN ACOUSTICS

J. Poblet-Puig, A.V. Shanin
Universitat Politècnica de Catalunya, Barcelona, Spain
M.V.Lomonosov Moscow State University, Moscow, Russia
What we would like to solve?

An external diffraction problem (stationary or non-stationary) with Neumann boundary conditions and some sources
The main dilemma: FEM or BEM?

FEM: Finite Elements Method

Equation solved:
\[ \Delta p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = f(r, t) \]

Difficulties:

1. Too many nodes
2. Something should be done at infinity (evacuation of waves), say PML
BEM: Boundary Elements Method

Equation solved:

\[
\frac{1}{2} u(r) = \int_s \left[ G(r, r') f(r') - u(r') \partial_n G(r, r') \right] ds + F(r)
\]

(Direct Kirchhoff formulation, after application Green’s theorem to Helmholtz equation)

Benefits:
1. No problems at infinity
2. Relatively low amount of nodes
3. No dispersion

Problems:
1. Singular and oversingular integrals, problems with accuracy
2. Spurious resonances (painful artifacts, see below)
3. Dense matrices
Spurious frequencies (in BEM) and how to kill them

The problem we solve

on spurious frequencies the matrices become ill-conditioned

Spurious frequencies are the eigenvalues of the dual problem

The remedy: CFIE (Burton-Miller) approach

\[ \frac{1}{2} u(r) = \int_s \left[ G(r, r') f(r') - u(r') \partial_n G(r, r') \right] ds \]

\[ + \frac{1}{2} f(r) = \int_s \left[ \partial_n G(r, r') f(r') - u(r') \partial^2_{n n} G(r, r') \right] ds \]

\[ \frac{1}{2} u(r) + \int_s \left[ u(r') \partial_n G(r, r') - \nu u(r') \partial^2_{n n} G(r, r') \right] ds = \frac{\nu}{2} f(r) + \int_s \left[ G(r, r') f(r') - \nu \partial_n G(r, r') f(r') \right] ds \]
New method. **Step 1: LEGO-BEM** (BAE – Boundary Algebraic Equations)

The main problem of BEM comes from its continuous nature.

If we consider discrete formulation from the very beginning, the Green’s function is not singular (and there should be no integration)

Our aim is to convert Burton-Miller (CFIE) technique to the discrete form

The idea itself appears to be not new 😎

But we were the first to implement an analog of CFIE!

C. Saltzer, 1958;
P-G. Martinsson et al, 2009, 2010
An example of LEGO mesh
Formalism of the method

Equation to solve: \[ \Delta u + k^2 u = f \]

Approximate form: \[ \beta_{i,j} u_j = f_i \]

Green's function: \[ \beta_{i,j} G_{j,m} = \delta_{m,i} \]

Split \( \beta_{i,j} \) between inner and outer mesh:

\[ \beta_{i,j} = \beta_{i,j}^I + \beta_{i,j}^O \]

Both should be symmetrical:

\[ \beta_{i,j}^I = \beta_{j,i}^I \quad \beta_{i,j}^O = \beta_{j,i}^O \]

Example: 5 point scheme non-zero values of \( \beta_{i,j} \)

\[ -4 + k^2 h^2 \]

\[ +1 \]
FEM equation we solve:

$$\sum_{j \in \Omega} \beta_{i,j}^O u_j = f_i, \quad f_i \neq 0 \quad \text{only for} \quad i \in \partial \Omega$$

in operator form:

$$\beta^{oo} u^o = f^o$$

BAE (LEGO-BEM) formulation of the problem

$$\sum_{i \in \Omega} \sum_{j \in \partial \Omega} G_{m,i} \beta_{i,j}^O u_j = \sum_{j \in \partial \Omega} G_{m,j} f_j \quad m \in \partial \Omega$$

in operator form:

$$G^{BO} \beta^{OB} u^B = G^{BB} f^B$$

It may be not clear from the first glance, but this is a direct analogue of the Kirchhoff BEM formulation!
Interpretation of the boundary equation as the NtD (Neumann to Dirichlet) operator

\[ G^{BO} \beta^{OB} u^B = G^{BB} f^B \]

\[ f^B = \beta^{BO} u^O \]

Neumann boundary operator, discrete analogue of \( \partial_n u \)

\[ u^B = (G^{BO} \beta^{OB})^{-1} G^{BB} (\beta^{BO} u^O) \]

Dirichlet boundary data

NtD operator

Neumann boundary data

This relation guarantees that there are no waves coming from infinity!
The price we pay: computation (tabulation) of the Green’s function

\[ G(m_1, m_2) = \frac{\hbar^2}{4\pi i} \int_{-\pi}^{\pi} \frac{\exp\{im_1 \xi + i|m_2|\Xi(\xi)\}}{\Xi(\xi)} d\xi \]

\[ \Xi(\xi) = \text{arccos}(2 - \cos \xi - \hbar^2 k^2 / 2) \]
In principle, we can use BAE, but the accuracy is poor due to rough ("LEGO") boundary modeling!
New method, **Step 2:** Combining FEM with BAE

We apply BAE to $\Omega_{\text{ext}}$ to find the link between Neumann and Dirichlet data on $\Gamma_{\text{ext}}$.

Then we apply FEM to $\Omega_{\text{int}}$ using BAE as a boundary condition on $\Gamma_{\text{ext}}$. 

---

$\Omega_{\text{int}}$ is the inner domain, $\Omega_{\text{ext}}$ is the outer domain, $\Gamma_{\text{int}}$ is the inner boundary, and $\Gamma_{\text{ext}}$ is the outer boundary.
Formalism of the method

\[ I : \Omega_{\text{int}} \]
\[ O : \Omega_{\text{ext}} \]
\[ i : \Omega_{\text{int}} \setminus \Gamma_{\text{ext}} \]
\[ B : \Gamma_{\text{ext}} \]

**FEM on** \( \Omega_{\text{int}} \):

\[ \beta^{il} u^I = f^i \quad (*) \]

**Boundary condition on** \( \Gamma_{\text{ext}} \):

Note that \( (\beta^{BO} + \beta^{BI}) u = 0 \) on \( \Gamma_{\text{ext}} \) (all sources are on \( \Gamma_{\text{int}} \))

thus \( \beta^{BO} u^O = -\beta^{BI} u^I \) (Neumann boundary data)

\[ u^B = - (G^{BO} \beta^{OB})^{-1} G^{BB} (\beta^{BI} u^I) \quad (**) \]

System \((*) , (**)) is what we solve!
Example of mesh $\Omega_{\text{int}}$