A simple robust and accurate a posteriori subcell finite volume limiter for the discontinuous Galerkin method

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Objectives

(1) Design a new limiter for the discontinuous Galerkin finite element method that is **simple**, **robust** and **accurate**

(2a) The new limiter **must not destroy** the subcell resolution capability of the DG scheme, neither at discontinuities, nor in smooth regions, where it might have been erroneously activated, or, equivalently

(2b) The limiter must act on a characteristic length scale of $h/(N+1)$ and **not** on the length scale $h$ of the main grid, i.e. accuracy improves with $N$ **even at shocks**

(3) The DG limiter should **not** contain **problem-dependent parameters**, like, e.g., the well-known parameter $M$ of the classical TVB limiter of Cockburn and Shu.

(4) The new limiter should work well for **very high** polynomial degrees, say $N=9$.

(5) Ideally, the final DG scheme should become **as robust** as a traditional **second order TVD finite volume scheme**, but **more accurate** on a given computational mesh of characteristic mesh size $h$
Unlimited Fully Discrete One-Step ADER-DG Scheme

Governing hyperbolic PDE system of the form

\[ \frac{\partial Q}{\partial t} + \nabla \cdot F(Q) = 0 \]  \hspace{1cm} (PDE)

with the vector of conserved variables \( Q \) and the nonlinear flux tensor \( F(Q) \).

The discrete solution at time \( t^n \) is represented by piecewise polynomials of degree \( N \) over spatial control volumes \( T_i \) as

\[ u_h(x, t^n) = \sum_l \Phi_l(x) \hat{u}^n_l, \quad x \in T_i \]  \hspace{1cm} (DG)

Multiplication with a test function \( \phi_k \) from the space of piecewise polynomials of degree \( N \) and integration over a space-time control volume \( T_i \times [t^n, t^{n+1}] \) yields:

\[ \int_{t^n}^{t^{n+1}} \int_{T_i} \Phi_k \frac{\partial Q}{\partial t} dx dt + \int_{t^n}^{t^{n+1}} \int_{\partial T_i} \Phi_k F(Q) \cdot n dS dt - \int_{t^n}^{t^{n+1}} \int_{T_i} \nabla \Phi_k \cdot F(Q) dx dt = 0 \]
Unlimited Fully Discrete One-Step ADER-DG Scheme

We then introduce the discrete solution (DG) and an **element-local space-time predictor** \( q_h(x,t) \), together with a classical (monotone) numerical flux \( G \), as it is used in Godunov-type finite volume schemes.

The fully discrete one-step ADER-DG scheme then simply reads:

\[
\left( \int_{T_i} \Phi_k \Phi_l d\mathbf{x} \right) \left( \hat{u}^{n+1}_l - \hat{u}^n_l \right) + \int_{t^n}^{t^{n+1}} \int_{T_i} \Phi_k G(q^-_h, q^+_h) \cdot \mathbf{n} dS dt - \int_{t^n}^{t^{n+1}} \int_{T_i} \nabla \Phi_k \cdot \mathbf{F}(q_h) d\mathbf{x} dt = 0
\]

But how to compute the space-time predictor \( q_h(x,t) \), since at the beginning of a time step, only the discrete spatial solution \( u_h(x,t^n) \) at time \( t^n \) is known?

Use a weak integral form of the PDE in space-time and solve an element-local Cauchy problem *in the small*, with initial data \( u_h(x,t^n) \), similar to the MUSCL-Hancock scheme or the ENO scheme of Harten et al.
Element-local Space-time Predictor

Rewrite the governing PDE in a reference coordinate system $\xi-\tau$ on a reference element $T_E$:

$$
\frac{\partial Q}{\partial \tau} + \nabla_\xi \cdot F^*(Q) = 0, \quad F^* := \Delta t (\frac{\partial \xi}{\partial x})^T \cdot F(Q).
$$

We introduce the two space-time integral operators

$$
\langle f, g \rangle = \int_0^1 \int_{T_E} (f(\xi, \tau) \cdot g(\xi, \tau)) d\xi d\tau, \quad [f, g]^\tau = \int_{T_E} (f(\xi, \tau) \cdot g(\xi, \tau)) d\xi.
$$

The discrete space-time predictor solution and the discrete flux are defined as

$$
q_h = q_h(\xi, \tau) = \sum_l \theta_l(\xi, \tau) \hat{q}_l := \theta_l \hat{q}_l, \quad \text{nodal space-time basis } \theta_l
$$

$$
F^*_h = F^*_h(\xi, \tau) = \sum_l \theta_l(\xi, \tau) \hat{F}^*_l := \theta_l \hat{F}^*_l, \quad \hat{F}^*_l = F^*(\hat{q}_l).
$$
Element-local Space-time Predictor

Multiplication with a space-time test function and integration over the space-time reference element \( T_E \times [0,1] \) yields:

\[
\left\langle \theta_k, \frac{\partial q_h}{\partial \tau} \right\rangle + \left\langle \theta_k, \nabla_{\xi} \cdot F^*_h(q_h) \right\rangle = 0.
\]

The initial condition \( u_h(x,t^n) \) is introduced in a weak sense after integration by parts in time (upwinding in time, causality principle):

\[
[\theta_k, q_h]^1 - [\theta_k, u_h]^0 - \left\langle \frac{\partial}{\partial \tau} \theta_k, q_h \right\rangle + \left\langle \theta_k, \nabla_{\xi} \cdot F^*_h \right\rangle = 0.
\]

The above element-local nonlinear system is easily solved via the following fast-converging fixed-point iteration (discrete Picard iteration):

\[
\left( [\theta_k, \theta_l]^1 - \left\langle \frac{\partial}{\partial \tau} \theta_k, \theta_l \right\rangle \right) \hat{q}_l^{r+1} = [\theta_k, \Phi_l]^0 \hat{u}_l^n - \left\langle \theta_k, \nabla_{\xi} \theta_l \right\rangle \cdot F^*(\hat{q}_l^r),
\]
A new *a posteriori* limiter of DG-FEM methods

- Motivation: develop a **simple, robust** and **parameter-free** limiter for DG that **always works** and which does **not destroy** the **subcell resolution** of DG.

- Conventional DG limiters use either artificial viscosity, which needs parameters to be tuned, or nonlinear FV-type reconstruction/limiters (TVB, WENO, HWENO), which **usually destroy** the subcell resolution properties.

- Our new approach: extend the successful *a posteriori* MOOD method of Loubère et al., developed in the FV context, also to the DG-FEM framework.

- As very simple *a posteriori* detection criteria, we only use
  - A relaxed discrete maximum principle (**DMP**) in the sense of polynomials
  - **Positivity** of the solution and absence of floating point errors (**NaN**)

- If one of these criteria is violated after a time step, the scheme **goes back** to the old time step and **recomputes** the solution in the troubled cells, using a more robust ADER-WENO or TVD FV scheme on a **fine subgrid** composed of **2N+1** subcells per space dimension.
A new \textit{a posteriori} limiter of DG-FEM methods

- Classical DG limiters, like WENO/HWENO/slope/moment limiters are based on \textbf{nonlinear data post-processing}, while the new DG limiter \textbf{recomputes} the discrete solution with a more robust scheme, starting again from a \textbf{valid solution} available at the old time level.

- Alternative description: dynamic, element-local \textbf{checkpointing} and \textbf{restarting} of the solver with a more robust scheme on a finer grid.

- This enables the limiter even to \textbf{cure} floating point errors (\textbf{NaN} values appearing after division by zero or after taking roots of negative numbers).

- The new method is by construction \textbf{positivity preserving}, if the underlying finite volume scheme on the subgrid preserves positivity.

- \textbf{Local limiter} (in contrast to WENO limiters for DG), since it requires only information from the cell and its direct neighborhood.

- As \textbf{accurate} as a high order \textbf{unlimited DG scheme} in smooth flow regions, but at the same time as \textbf{robust} as a \textbf{second order TVD scheme} at shocks or other discontinuities, but also at strong rarefactions.
Classical TVB slope/moment limiting of DG

If a classical nonlinear reconstruction-based DG limiter is activated erroneously, there may be important physical information that is lost forever!
A new \textit{a posteriori} limiter of DG-FEM methods

DG polynomials of degree $N=8$ (left) and \textbf{equivalent} data representation on $2N+1=17$ \textbf{subcells} (right). Arrows indicate projection (red) and reconstruction (blue)

\[ \mathcal{R} \circ \mathcal{P} = \mathcal{I} \]

We use $2N+1$ subcells to \textbf{match} the DG time step ($\text{CFL}<1/(2N+1)$) on the coarse grid with the FV time step ($\text{CFL}<1$) on the fine subgrid.
A new *a posteriori* limiter of DG-FEM methods

Projection from the DG polynomials to the subcell averages

\[ \mathbf{v}_{i,j}^n = \frac{1}{|S_{i,j}|} \int_{S_{i,j}} \mathbf{u}_h(\mathbf{x}, t^n) d\mathbf{x} = \frac{1}{|S_{i,j}|} \int_{S_{i,j}} \phi_l(\mathbf{x}) d\mathbf{x} \hat{\mathbf{u}}_l^n, \quad \forall S_{i,j} \in S_i. \]

Reconstruction of DG polynomials from the subcell averages

\[ \int_{S_{i,j}} \mathbf{u}_h(\mathbf{x}, t^n) d\mathbf{x} = \int_{S_{i,j}} \mathbf{v}_h(\mathbf{x}, t^n) d\mathbf{x}, \quad \forall S_{i,j} \in S_i. \]

\[ \int_{T_i} \mathbf{u}_h(\mathbf{x}, t^n) d\mathbf{x} = \int_{T_i} \mathbf{v}_h(\mathbf{x}, t^n) d\mathbf{x}. \quad \text{Linear constraint: conservation} \]

Overdetermined system, solved by a constrained LSQ algorithm.
A posteriori detection criteria and DG-MOOD flowchart

Positivity:

$$\prod_k\left(u_h^\ast(x, t^{n+1})\right) > 0.$$  

Relaxed DMP in the sense of polynomials:

$$\min_{y \in \mathcal{V}_i}(u_h(y, t^n)) - \delta \leq u_h^\ast(x, t^{n+1}) \leq \max_{y \in \mathcal{V}_i}(u_h(y, t^n)) + \delta,$$
DMP in the sense of polynomials

\[
\min_{y \in \mathcal{V}_i}(u_h(y, t^n)) - \delta \leq u_h^*(x, t^{n+1}) \leq \max_{y \in \mathcal{V}_i}(u_h(y, t^n)) + \delta,
\]

\[
\max_{y \in \mathcal{V}_i}(u_h(y, t^n))
\]

\[
\min_{y \in \mathcal{V}_i}(u_h(y, t^n))
\]

\[
u_h(x, t^n)
\]

\[
u_h^*(x, t^{n+1})
\]
Summary of the ADER-DG-MOOD scheme

Verification of the DMP and the positivity on the candidate solution $u_h^*(x, t^{n+1})$:

$$\min_{y \in \mathcal{V}_i}(v_h(y, t^n)) - \delta \leq v_h^*(x, t^{n+1}) \leq \max_{y \in \mathcal{V}_i}(v_h(y, t^n)) + \delta, \quad \forall x \in T_i,$$

$$\pi_k(u_h^*(x, t^{n+1})) > 0, \quad \forall x \in T_i, \forall k,$$

If a cell does not satisfy both criteria, flag it as troubled cell, $\beta_i^{n+1} = 1$, **discard** the DG solution and **recompute** it with a more robust third order ADER-WENO or an even more robust **second order TVD finite** volume scheme on the **fine subgrid**:

$$v_h(x, t^{n+1}) = A(v_h(x, t^n))$$

$$v_h(x, t^n) = \begin{cases} 
  P(u_h(x, t^n)) & \text{if } \beta_j^n = 0, \\
  A(v_h(x, t^{n-1})) & \text{if } \beta_j^n = 1. 
\end{cases} \quad x \in T_j \quad \forall T_j \in \mathcal{V}_i.$$

Finally, **reconstruct** the DG polynomial from the subcell averages:

$$u_h(x, t^{n+1}) = R(v_h(x, t^{n+1})) \quad \text{or} \quad u_h(x, t^{n+1}) = R(A(v_h(x, t^n)))$$
### 2D Numerical Convergence Results P2-P9 (Euler)

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ADER-DG-MOOD Results

Sod shock tube, 20x5 elements (N=9)

Limited cells (red), Unlimited cells (blue)
ADER-DG-MOOD Results

Lax shock tube,
20x5 elements \((N=9)\)

Limited cells (red),
Unlimited cells (blue)
A posteriori subcell finite volume limiting of the Discontinuous Galerkin method

ADER-DG-MOOD Results

Shock-density interaction problem of Shu & Osher
40x5 cells (N=9). Unlimited cells (blue) and limited cells (red)
Double Mach Reflection Problem

300x100 cells (N=2, 5, 9). Unlimited cells (blue) and limited cells (red)
3D Spherical Explosion Problem

100³ cells (N=9), corresponding to 10 billion space-time degrees of freedom per time step. Unlimited cells (blue) and limited cells (red)
Coupling of *a posteriori* subcell limiters for DG with AMR
Coupling of AMR with a posteriori subcell limiters for DG

ADER-DG (N=9) with a posteriori ADER-WENO subcell limiter and space-time adaptive mesh refinement (AMR) yields an unprecedented resolution of shocks and contact waves.
Coupling of AMR with a posteriori subcell limiters for DG

Double Mach reflection problem using ADER-DG (N=9) with a posteriori ADER-WENO subcell limiter and space-time adaptive mesh refinement (AMR)
Natural Extension to Unstructured Meshes

Subgrid for N=1 to N=6 in 2D
Natural Extension to Unstructured Meshes

Subgrid for N=1 to N=5 in 3D
Natural Extension to Unstructured Meshes

Circular explosion problem in 2D ($N=5$)
Natural Extension to Unstructured Meshes

Double Mach reflection problem in 2D ($N=4$)
Natural Extension to Unstructured Meshes

Spherical explosion problem in 3D (N=3)
Natural Extension to Unstructured Meshes

Mach 3 flow over a sphere \((N=3)\)
Conclusions

• New, simple robust and accurate *a posteriori* subcell finite volume limiter for the discontinuous Galerkin finite element method

• High order fully discrete one-step ADER time discretization

• Available for uniform and space-time adaptive (AMR) Cartesian grids as well as for general triangular and tetrahedral unstructured meshes

• The *a posteriori* MOOD framework of Loubère, Clain and Diot has been found to be an ideal framework to devise a simple and robust limiter for DG schemes

• Why *a posteriori*: It is much simpler to *observe* (and cure) the occurrence of a troubled cell rather than to *predict* (and avoid) its occurrence from given data.

• Element-local *checkpointing* and solver *restarting* is even able to *cure* floating point errors (*NaN*, e.g. after division by zero)

• Future extension: *Lagrangian-type DG schemes* on unstructured ALE meshes